Series

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Definition (Infinite series)

If $\{a_n\}$ is an infinite sequence ($a_n \in \mathbb{R}$ for $n = 1, 2, 3, \ldots$), then the expression of the form

$$a_1 + a_2 + a_3 + \ldots = \sum_{n=1}^{\infty} a_n$$

is called an infinite series, or simply a series.
Each number a_n is called nth term.

$$S_n = a_1 + a_2 + a_3 + \ldots + a_n = \sum_{k=1}^{n} a_k$$

is called nth partial sum.

Definition (convergence and divergence of series)

An infinite series

$$\sum_{n=1}^{\infty} a_n$$

with sequence of partial sums (S_n) is convergent (or converges) if $\lim_{n\to\infty} S_n = S$ for some real number S. The series is divergent (or diverges) if this limit does not exists.

If $\sum_{n=1}^{\infty} a_n$ is a convergent infinite series and $\lim_{n\to\infty} S_n = S$, then S is called the sum of the series and we write

$$\sum_{n=1}^{\infty} a_n \stackrel{def}{=} S$$

If a series diverges, it has no sum.

Prove convergance of the series making direct use of the definition of convergance

1. $\sum_{n=1}^{\infty} \frac{1}{n!}$;
2. $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Find the sum

1. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$;
2. $\sum_{n=1}^{\infty} \frac{1}{(3n-2)(3n+1)}$;
3. $\sum_{n=1}^{\infty} \ln \frac{n(n+2)}{(n+1)^2}$.

Theorem (necessary condition of convergence of series)

If an infinite series $\sum_{n=1}^{\infty} a_n$ *is convergent, then* $\lim_{n\to\infty} a_n = 0$.

Corollary

If $\lim_{n\to\infty} a_n \neq 0$, *then the infinite series* $\sum_{n=1}^{\infty} a_n$ *is divergent.*

Example

Show that the series are divergent

1. $\sum_{n=1}^{\infty} \frac{n+2}{n+100}$
2. $\sum_{n=1}^{\infty} n \sin \frac{1}{n}$

Definition (harmonic series)

We call the series of the form

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots.$$

the harmonic series.

Remark

The harmonic series is divergent.

Definition (geometric series)

A series of the form

$$\sum_{n=1}^{\infty} a_1 q^{n-1} = a_1 + a_1 q + a_1 q^2 + \ldots .$$

we call a geometric series with a quotient q.

Theorem (divergence of geometric series)

The geometric series

$$\sum_{n=1}^{\infty} a_1 q^{n-1} = a_1 + a_1 q + a_1 q^2 + \ldots$$

with $a_1 \neq 0$

1. *converges and has the sum* $\frac{a_1}{1-q}$ *if* $|q| < 1$
2. *diverges if* $|q| \geq 1$.

Definition (p-series, Dirichlet series)

A series of the form

$$\sum_{n=1}^{\infty} \frac{1}{n^p},$$

where p is positive real number is called p-series.

Theorem (divergence of p-series)

The p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p},$$

1. *converges if* $p > 1$,
2. *diverges if* $p \leq 1$

Theorem (convergence of linear combination)

Let the series $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ *be convergent and let* $\alpha, \beta \in \mathbb{R}$.

Then

$$\sum_{n=1}^{\infty}(\alpha a_n + \beta b_n) = \alpha \sum_{n=1}^{\infty} a_n + \beta \sum_{n=1}^{\infty} b_n.$$

Tests of convergence

1. Comparison Test.
2. Limit Comparison Test.
3. Ratio Test.
4. The Root Test.
5. The Integral Test.

Theorem (Comparison Test)

Let $0 \leq a_n \leq b_n$ for every $n \geq n_0$. Then:

1. *If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges,*
2. *If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.*

Example

Check convergence of the following series

1. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+1}$,
2. $\sum_{n=1}^{\infty} \frac{\operatorname{arc\,tg} n}{n^2}$.

Theorem (Limit Comparison Test)

Let $a_n, b_n > 0$ *for every* $n \geq n_0$ *and let*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = k,$$

where $0 < k < \infty$.

Then either both series $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *converge or both diverge to* ∞.

Example

Check convergence of the following series

1. $\sum_{n=1}^{\infty} \frac{2n-1}{3n^2-2n+1}$,
2. $\sum_{n=1}^{\infty} \frac{3^n-2^n}{4^n-3^n}$.

Theorem (Ratio Test, d'Alembert's Ratio Test)

Let

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = q.$$

Then the series $\sum_{n=1}^{\infty} a_n$ *converges if* $q < 1$ *and diverges if* $q > 1$.

Remark

If $q = 1$ then the series may be convergent or divergent.

Example

Check convergence of the following series

1. $\sum_{n=1}^{\infty} \frac{2^n}{n!}$,
2. $\sum_{n=1}^{\infty} \frac{2^n}{n^2}$,
3. $\sum_{n=1}^{\infty} \frac{2^n+3^n}{3^n+4^n}$.

Theorem (The Root Test, The Cauchy Root Test)

Let

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = q.$$

Then the series $\sum_{n=1}^{\infty} a_n$ *converges if* $q < 1$ *and diverges if* $q > 1$.

Remark

If $q = 1$ then the series may be convergent or divergent.

Example

Check convergence of the following series

1. $\sum_{n=1}^{\infty} \left(\frac{n+1}{n-1}\right)^n$,
2. $\sum_{n=1}^{\infty} \frac{n^{100}}{\pi^n}$.

Theorem (The Integral Test)

If the function f is positive valued, continuous, and decreasing in $[n_0, \infty)$, where $n_0 \in \mathbb{N}$. Then the series

$$\sum_{n=n_0}^{\infty} f(n)$$

and improper integral

$$\int_{n_0}^{\infty} f(x)dx$$

are both convergent or both divergent to ∞.

Example

Check convergence of the following series

1. $\sum_{n=1}^{\infty} \frac{1}{3n+1}$,
2. $\sum_{n=1}^{\infty} \frac{1}{n \ln n}$,
3. $\sum_{n=1}^{\infty} \frac{2n}{4n^2+9}$.

Definition (Alternating series)

A series of the form

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n,$$

where $a_n > 0$, is called an alternating series.

Theorem (Alternating Series Test)

If the sequence (a_n) is decreasing (non-increasing) and $\lim_{n \to \infty} a_n = 0$, then the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n$$

is convergent.

Definition (Absolutely convergent series)

An infinite series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent if the series

$$\sum_{n=1}^{\infty} |a_n|$$

is convergent.

Theorem

If an infinite series is absolutely convergent, then it is convergent.

Definition (Conditionally convergent series)

An infinite series is conditionally convergent if it is convergent and it is not absolutely convergent.

Example

Check convergence of the following series

1. $\sum_{n=1}^{\infty} \frac{(-1)^n}{3n+1}$,
2. $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n}$,
3. $\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!}$.

Power series

A power series in the variable x is a series of the form

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots = \sum_{k=0}^{\infty} a_k x^k$$

where the coefficients $a_0, a_1, a_2, \ldots$ are real or complex numbers.
Given the power series

$$\sum_{k=0}^{\infty} a_k x^k$$

with a given choice of coefficients $a_0, a_1, a_2, \ldots$, what values of x give us convergent series, and which values give divergent series?

Theorem

For a power series

$$\sum_{k=0}^{\infty} a_k x^k$$

there are three possibilities:

1. *The power series* $\sum_{k=0}^{\infty} a_k x^k$ *diverges for all* $x \neq 0$
2. *The power series* $\sum_{k=0}^{\infty} a_k x^k$ *converges for all values of* x
3. *There is a positive number* R *such that* $\sum_{k=0}^{\infty} a_k x^k$ *converges for all values of* x *with* $|x| < R$ *and diverges for all values of* x *with* $|x| > R$.

At first sight, this looks like a very useless result, because it doesn't answer the question of which values of x are allowed. However, it is a very useful result: it tells us what sort of behaviour we can expect, and what to look for in a power series.

So, given our theorem, how do we go about calculating R?

Theorem

Given the power series $\sum_{k=0}^{\infty} a_k x^k$, *suppose that one of the following limits exist:*

$$K = \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right|, \quad K = \lim_{k\to\infty} \sqrt[k]{|a_k|}.$$

Then the following is true:

1. *If* $K = 0$ *then the power series* $\sum_{k=0}^{\infty} a_k x^k$ *converges for all values of* x;
2. *If* $K > 0$, *then the radius of convergence* R *of the power series* $\sum_{k=0}^{\infty} a_k x^k$ *is* $R = \frac{1}{K}$
3. *If either of the limits*

$$K = \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right|, \quad K = \lim_{k\to\infty} \sqrt[k]{|a_k|}$$

fails to exist, then the power series $\sum_{k=0}^{\infty} a_k x^k$ *diverges for all values of* $x \neq 0$.

Find the redious of convergance

1. $\sum_{n=1}^{\infty} \left(\frac{5}{3}\right)^n (x+5)^n$,
2. $\sum_{n=1}^{\infty} \frac{(x-3)^n}{2^n}$,
3. $\sum_{n=1}^{\infty} 2^n (4-x)^n$.

Find the intervals of convergance of the power series

1. $\sum_{n=1}^{\infty} \frac{1}{n}(x-2)^n$,
2. $\sum_{n=1}^{\infty} \frac{(2x-3)^n}{4^{2n}}$.