Lecture

Multi-variable functions

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Definition (plane, space, $\mathbb{R}^n$)

$$\mathbb{R}^2 = \{(x, y); \quad x, y \in \mathbb{R}\}$$

$$\mathbb{R}^3 = \{(x, y, z); \quad x, y, z \in \mathbb{R}\}$$

$$\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n); \quad x_i \in \mathbb{R}, i = 1, 2, \ldots, n\}$$

Definition (distance of points)

$$d(P_1, P_2) = |P_1 P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$d(P_1, P_2) = |P_1 P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

$$d(P_1, P_2) = |P_1 P_2| = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2 + \ldots + (y_n - x_n)^2}$$

Definition (open ball)

The open (metric) ball of radius $r > 0$ centred at a point P_0 is defined by

$$B(P_0, r) = \{P : \quad d(P, P_0) < r\}$$

In $\mathbb{R}^2$ an open ball is an open disk.

Definition (neighbourhood)

A set V is a neighbourhood of a point P if there exists an open ball with centre P and radius $r > 0$, such that $B(P, r) = \{x \in \mathbb{R}^n \mid d(x, P) < r\}$ is contained in V.

Definition (punctured neighbourhood)

A punctured neighbourhood of a point P (sometimes called a deleted neighbourhood) is a neighbourhood of P, without $\{P\}$.

Definition (bounded set)

If exists P_0 and a number $r > 0$ such that the set A is contained in the ball $B(P_0, r)$, then the set A is called bounded set.
In the opposite case the set A is called unbounded.

Definition (Interior point of a set, Interior of a set)

If there exists an open ball with centre P contained in the set A, then P is called interior point of the set A.
The set of all interior point of the set is called the interior of the set.

Definition (open set)

If every point of a set is its interior point, then the set is called an open set.

Definition (boundary)

If every ball with centre P contains points belonging to the set A and points not belonging to the set (belonging to the complement of the set A), then P is called a boundary point of the set A.
The set of all boundary points is called the boundary of a set.

Definition (closed set)

If a set contains its boundary then it is called a closed set.

Definition (domain, closed domain)

Nonempty subset of $\mathbb{R}^n$ is called a domain, if:

1. it is open
2. cannot be represented as the union of two or more disjoint nonempty open sets

A domain with its boundary is called a closed domain.

Definition (Functions of Two Variables)

Let $A \subset \mathbb{R}^2$. A function f of two variables is a rule that assigns to each ordered pair (x, y) in A a unique real number denoted by $f(x, y)$. The set A is called the domain of f and its range is the set of values that f takes on, i.e., $\{f(x, y) | \quad (x, y) \in A\}$.

Notation

$$f : A \to \mathbb{R}^2$$

We often write $z = f(x, y)$ to make explicit the value taken on by f at the point (x, y). The variables x and y are independent variables and z is the dependent variable.

Definition (Functions of Three Variables)

A function of three variables, f, is a rule that assigns to each ordered triple (x, y, z) in a domain $A \subset \mathbb{R}^3$ a unique real number denoted by $f(x, y, z)$.
Notation

$$f : A \to \mathbb{R}^2$$

or $u = f(x, y, z)$, where $(x, y, z) \in A$.

Definition (Functions of n Variables)

A function of n variables is a rule that assigns a number $z = f(x_1, \dots, x_n)$ to an n-tuple $(x_1, \dots, x_n)$ of real numbers.

Example

For example, if a company uses n different ingredients in a food product, c_i is the cost per unit of the ith ingredient, and x_i is the units of the ith ingredient, then the total cost C of the ingredients is a function of n variables $x_1, \dots, x_n$:

$$C = f(x_1, \dots, x_n) = c_1 x_1 + \dots + c_n x_n$$

We can sometimes write functions more compactly with vector notation. If $x = [x_1, \dots, x_n]$, we may write $f(x)$ in place of $f(x_1, \dots, x_n)$. So we could write the cost function as

$$f(x) = c \cdot x$$

where $c = [c_1, \dots, c_n]$.

Example

Find and plot the domain of the following functions

1. $f(x, y) = \frac{1}{\sqrt{x}} + \sqrt{y}$
2. $f(x, y) = \frac{1}{\sqrt{1-x^2-y^2}}$

Definition (sequence of points in $\mathbb{R}^2$)

A sequence of points in $\mathbb{R}^2$ we call a mapping that assigns each natural number a point of plane.
We denote such sequence by (P_n), where $P_n = (x_n, y_n)$ is nth element of the sequence. The set of all elements $\{(x_n, y_n);\ n \in \mathbb{N}\}$ is denoted by $\{P_n\}$ or $\{(x_n, y_n)\}$.

Definition (Proper limit)

$$\lim_{n\to\infty} P_n = P_0 \Leftrightarrow (\lim_{n\to\infty} x_n = x_0 \wedge \lim_{n\to\infty} y_n = y_0)$$

Remark

A sequence (P_n) is convergent to a point P_0, if in every ball with centre P_0 there are almost all elements of the sequence.

Definition (Heine's definition of a function limit)

Let $(x_0, y_0) \in \mathbb{R}^2$ and the function f be defined at least in the punctured neighbourhood $S(x_0, y_0)$ of (x_0, y_0). The number g is called proper limit of function f at point (x_0, y_0) denoted by

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = g$$

if and only if

$$\underset{\substack{(x_n, y_n) \\ \{(x_n, y_n)\} \subset S(x_0, y_0)}}{\forall} \left(\lim_{n\to\infty} (x_n, y_n) = (x_0, y_0) \right) \Rightarrow \left(\lim_{n\to\infty} f(x_n, u_n) = g \right)$$

Remark

Improper limit we define in the same way.

Definition (Cauchy's definition of a function limit)

Let f be a function of two variables defined on a disk with centre (x_0, y_0), except possibly at (x_0, y_0). Then we say that the limit of $f(x, y)$ as (x, y) approaches (x_0, y_0) is L and we write

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

if for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that $|f(x, y) - L| < \varepsilon$ whenever $0 < d((x_0, y_0), (x, y)) < \delta$

This means that the values of $f(x, y)$ can be made as close as we wish to the number L by taking the point (x, y) close enough to the point (x_0, y_0).

Theorem (Arithmetic of limits)

If functions f and g have proper limits at point (x_0, y_0), then

1. $$\lim_{(x,y)\to(x_0,y_0)}[f(x,y)+g(x,y)] = \lim_{(x,y)\to(x_0,y_0)} f(x,y) + \lim_{(x,y)\to(x_0,y_0)} g(x,y)$$
2. $$\lim_{(x,y)\to(x_0,y_0)}[f(x,y)\cdot g(x,y)] = \lim_{(x,y)\to(x_0,y_0)} f(x,y) \cdot \lim_{(x,y)\to(x_0,y_0)} g(x,y)$$
3. $$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y)}{g(x,y)} = \frac{\lim_{(x,y)\to(x_0,y_0)} f(x,y)}{\lim_{(x,y)\to(x_0,y_0)} g(x,y)}, \text{ if } \lim_{(x,y)\to(x_0,y_0)} g(x,y) \neq 0$$

Theorem (limit of composite function)

If functions p, q and f satisfy the following conditions

1. $\lim\limits_{(x,y)\to(x_0,y_0)} p(x,y) = p_0, \quad \lim\limits_{(x,y)\to(x_0,y_0)} q(x,y) = q_0$
2. $(p(x,y), q(x,y) \neq (p_0, q_0))$ *for every* $(x,y) \in S(p_0, q_0)$
3. $\lim\limits_{(x,y)\to(x_0,y_0)} f(p,q) = g$

then

$$\lim_{(x,y)\to(x_0,y_0)} f(p(x,y), q(x,y)) = g$$

Remark

We can admit improper limits in both theorems, if results are well defined.

We have no l'Hospital's rule to calculate limits of indefinite terms of multivalued functions.

Example

Calculate limits if exist

1. $$\lim_{(x,y)\to(1,2)} \frac{x^2+y}{2x^2+y^3}$$
2. $$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^3+y^3}$$

Definition (Continuity)

Let $(x_0, y_0) \in \mathbb{R}^2$ and let the function f be defined on a disk $O(x_0, y_0)$ with centre (x_0, y_0). The function f is called continuous at point (x_0, y_0) if and only if

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0, y_0)$$

Theorem (continuity of sum, product and quotient of functions)

If functions f and g are continuous at point (x_0, y_0)*, then at this point are also continuous functions:*

1. $f + g$
2. $f \cdot g$
3. $\frac{f}{g}$, *if only* $g(x_0, y_0) \neq 0$

Theorem (Continuity of composite function)

If the function p, q and f satisfy the following conditions

1. p *and* q *are continuous at point* (x_0, y_0)
2. f *is continuous at point* $(p_0, q_0) = (p(x_0, y_0), q(x_0, y_0))$

then the function $f(p(x, y), q(x, y))$ *is continuous at the point* (x_0, y_0).

Definition (Partial Derivatives of first order)

Let the function f be defined on a disk $O(x_0, y_0)$ with centre (x_0, y_0). The partial derivative of the first order of $f(x, y)$ with respect to x at the point (x_0, y_0) is

$$\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}.$$

The partial derivative of $f(x, y)$ with respect to y at the point (x_0, y_0) is

$$\frac{\partial f}{\partial y}(x_0, y_0) = \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y}.$$

Definition (Partial Derivatives on an open set)

If a function f has partial derivatives of first order at every point of an open set $D \subset \mathbb{R}^2$, then functions

$$\frac{\partial f}{\partial x}(x,y),\ \frac{\partial f}{\partial y}(x,y),\ \text{where } (x,y) \in D,$$

are called partial derivatives of first order on the set D and are denoted by $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ or f_x, f_y.

Remark

The definition of the partial derivatives for functions of more than two independent variables are analogous to the two variable definitions.

Example

Find the values of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$:

1. $f(x, y) = 2x^3 + y - 3xy^2 - 1$
2. $f(x, y) = \frac{2x^2}{y} - \frac{y^2}{x}$
3. $f(x, y) = x^y$
4. $f(x, y) = e^{-\cos x} \sin y$

Theorem (derivative of composite function (case 1))

Suppose that

1. $x = x(t)$, $y = y(t)$ *are both differentiable functions at* t_0,
2. $x = f(x, y)$ *has continuous partial derivatives at* $(x(t_0), y(t_0))$

Then composite function $F(t) = f(x(t), y(t))$ *is differentiable functions at* t_0

$$\frac{dF}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}.$$

Derivatives $\frac{dx}{dt}$, $\frac{dy}{dt}$ *are evaluated at* t_0, *and partial derivatives* $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ *at* $(x(t_0), y(t_0))$.

Example

1. If $z = x^2y + xy^3$, where $x = \cos t$, $y = \sin t$, find dz/dt when $t = \pi/2$.
2. Find dz/dt if $z = \sqrt{x^2 + y^2}$ and $x = e^{2t}$ and $y = e^{-2t}$.

Theorem (derivative of composite function (case 2))

Suppose that

1. $x = x(u, v)$, $y = y(u, v)$ *have partial derivatives at* (u_0, v_0),
2. $x = f(x, y)$ *has continuous partial derivatives at* $(x(u_0, v_0), y(u_0, v_0))$

Then the composite function $F(u, v) = f(x(u, v), y(u, v))$ *has at* (u_0, v_0) *partial derivatives*

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial u}, \quad \frac{\partial F}{\partial v} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial v}$$

Partial derivatives $\frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}, \frac{\partial y}{\partial u}, \frac{\partial y}{\partial v}$ *are evaluated at* (u_0, v_0), *and partial derivatives* $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ *at* $(x(u_0, v_0), y(u_0, v_0))$.

Example

Find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$ for the following examples

1. $z = e^{xy}\sin x$, where $x = 2s + 4t$, $y = \frac{2s}{3t}$.
2. $z = \ln(x^2 + y^2)$, where $x = e^s \cos t$ and $y = e^s \sin t$.
3. $w = xy + xz + yz$, where $x = st$, $y = e^{st}$, $z = x + t$.

Definition (Partial Derivatives of second order)

Let a function f has partial derivatives of first order $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ be defined on a disk $O(x_0, y_0)$ with centre (x_0, y_0).Partial Derivatives of second order of the function f at point (x_0, y_0) are defined as:

$$\frac{\partial^2 f}{\partial x^2}(x_0, y_0) = \left(\frac{\partial}{\partial x}\frac{\partial f}{\partial x}\right)(x_0, y_0), \quad \frac{\partial^2 f}{\partial y \partial x}(x_0, y_0) = \left(\frac{\partial}{\partial y}\frac{\partial f}{\partial x}\right)(x_0, y_0)$$

$$\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) = \left(\frac{\partial}{\partial x}\frac{\partial f}{\partial y}\right)(x_0, y_0), \quad \frac{\partial^2 f}{\partial y^2}(x_0, y_0) = \left(\frac{\partial}{\partial y}\frac{\partial f}{\partial y}\right)(x_0, y_0)$$

Definition (Partial Derivatives on an Open Set)

If a function f has partial derivatives of second order at every point of an open set $D \subset \mathbb{R}^2$, then functions $\frac{\partial^2 f}{\partial x^2}(x,y)$, $\frac{\partial^2 f}{\partial x \partial y}(x,y)$, $\frac{\partial^2 f}{\partial y \partial x}(x,y)$, $\frac{\partial^2 f}{\partial y^2}(x,y)$, where $(x,y) \in D$ are called partial derivatives of second order on the set D and are denoted by $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial x \partial y}$, $\frac{\partial^2 f}{\partial y \partial x}$, $\frac{\partial^2 f}{\partial y^2}$ or f_{xx}, f_{xy}, f_{yx}, f_{yy}.

Example

Calculate second order partial derivatives

1. $f(x,y) = x^2y - 2y^3x^2 + x - y - 1$
2. $f(x,y) = x \sin y$
3. $f(x,y) = \ln(x^2y - y^2)$

Example

Show that

1. $xz_x - z_y = 0$ if $z = xe^y$
2. $z_x + z_y = 1$ if $z = \ln(e^x + e^y)$

Theorem (Schwartz theorem)

If partial derivatives $\frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x}$ *are continuous at a point* (x_0, y_0), *then they are equal i.e.*

$$\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) = \frac{\partial^2 f}{\partial y \partial x}(x_0, y_0)$$

Remark

Analogous equalities are also true for mixed derivatives of n variable functions ($n \geq 2$), and mixed derivatives of higher order.