Lecture

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Definition (Local Minimum)

1. A function f has a local minimum at (x_0, y_0) if

$$f(x, y) \geq f(x_0, y_0),$$

for all domain points (x, y) in an open disk centered at (x_0, y_0).

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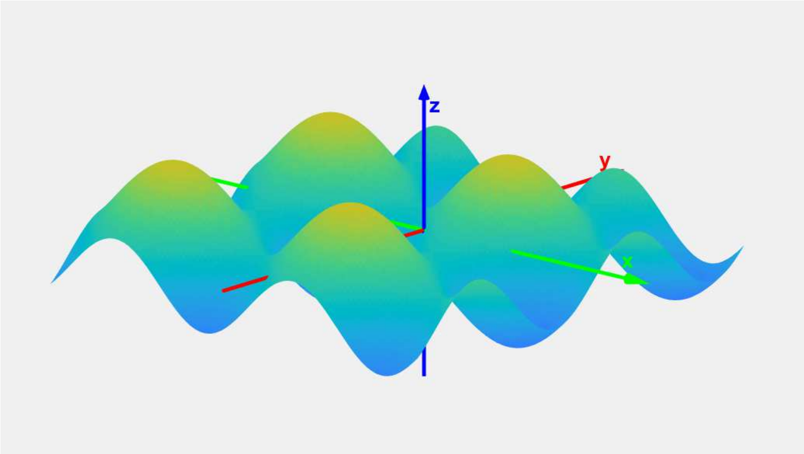
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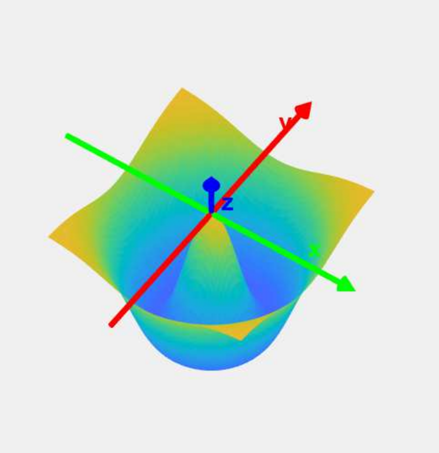
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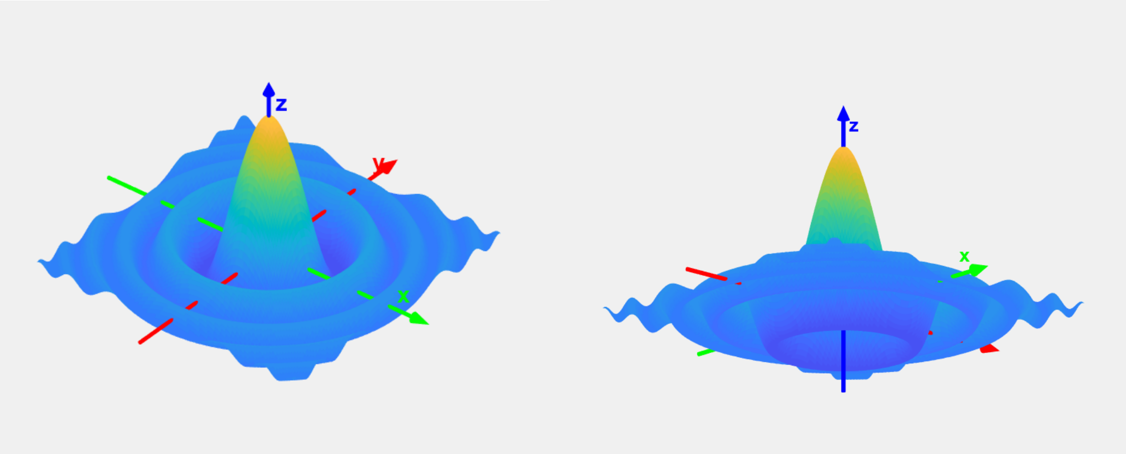
z
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Theorem (necessary condition)

If f satisfies the following conditions:

1. *has a local maximum or minimum at (x_0, y_0),*
2. *the first-order partial derivatives $\frac{\partial f}{\partial x}(x_0, y_0)$, $\frac{\partial f}{\partial y}(x_0, y_0)$ exist*

then

$$\frac{\partial f}{\partial x}(x_0, y_0) = 0, \quad \frac{\partial f}{\partial y}(x_0, y_0) = 0.$$

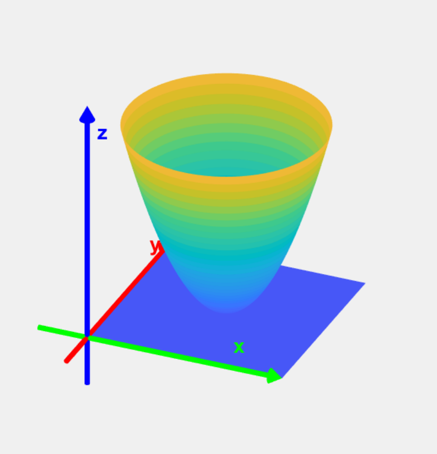
Definition

A point (x_0, y_0) is called a critical point (or stationary point) of f if $\frac{\partial f}{\partial x}(x_0, y_0) = 0, \quad \frac{\partial f}{\partial y}(x_0, y_0) = 0.$

Example

Find the critical points and extreme values of the function

$$f(x,y) = x^2 + y^2 - 4x - 4y + 10.$$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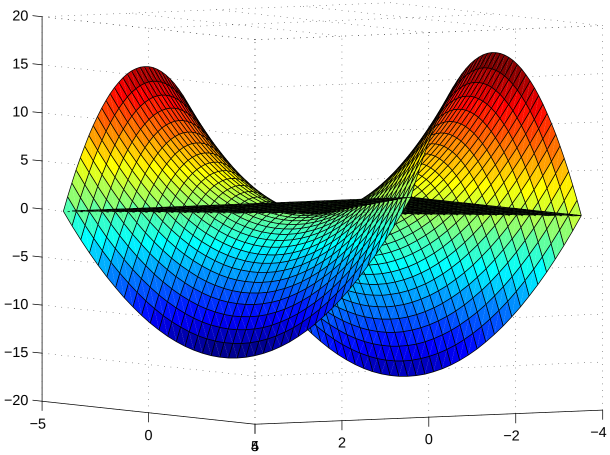
Example

Find the extreme values of $f(x, y) = y^2 - x^2$.


20
15
10
5
0
−5
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−20
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0
2
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−4

Theorem (Second Derivative Test)

Suppose the second partial derivatives of f are continuous on a disk with center (x_0, y_0) and let

1. $\frac{\partial f}{\partial x}(x_0, y_0) = 0, \quad \frac{\partial f}{\partial y}(x_0, y_0) = 0,$
2. $D = \det \begin{bmatrix} \frac{\partial^2 f}{\partial x^2}(x_0, y_0) & \frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) \\ \frac{\partial^2 f}{\partial y \partial x}(x_0, y_0) & \frac{\partial^2 f}{\partial y^2}(x_0, y_0) \end{bmatrix} > 0$

Then at (x_0, y_0) function f has a local extremum:

- *a local minimum, if* $\frac{\partial^2 f}{\partial x^2}(x_0, y_0) > 0$
- *a local maximum, if* $\frac{\partial^2 f}{\partial x^2}(x_0, y_0) < 0$.

If $D < 0$, then $f(x_0, y_0)$ is not a local maximum or minimum. In this case the point (x_0, y_0) is called a saddle point of f.

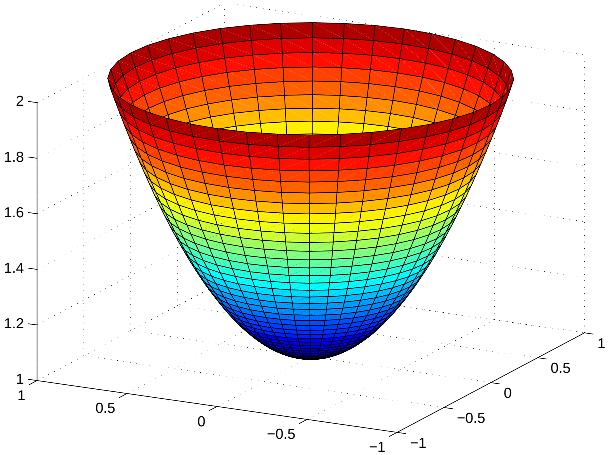


Figure: $f(x, y) = x^2 + y^2$

Fact

If $D = 0$, the test gives no information: f could have a local maximum or local minimum at (x_0, y_0), or (x_0, y_0) could be a saddle point of f.

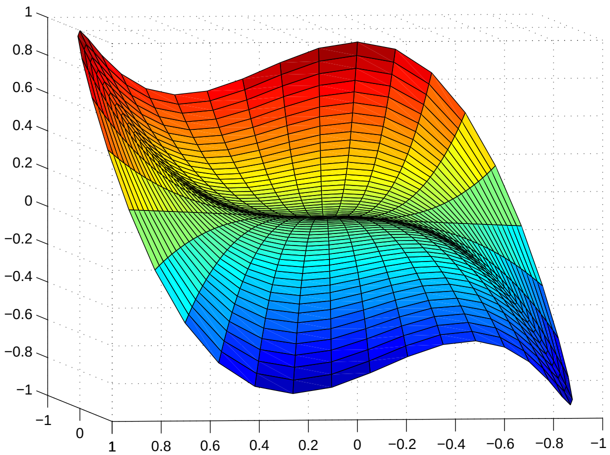


Figure: $f(x, y) = x^3 + y^3$

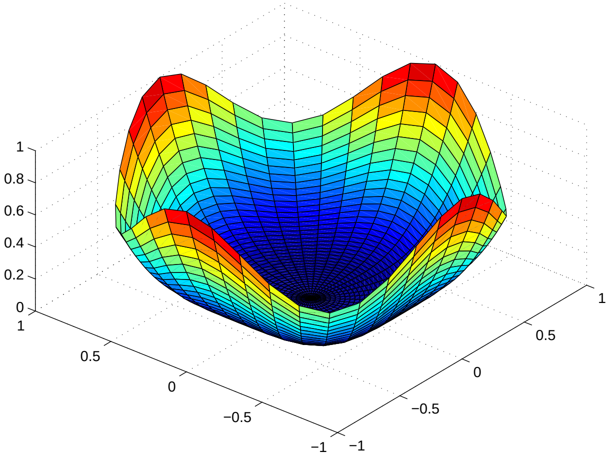


Figure: $f(x, y) = x^4 + y^4$

Example

Find the local maximum and minimum values and saddle points of

1. $f(x, y) = x^2 + 2y^2 - 4x + 4x$,
2. $f(x, y) = x^3 + xy^2 + 6xy$,
3. $f(x, y) = x^4 + y^4 - 4xy + 1$.

Absolute Maximum and Absolute Minimum

Definition

If

$$f(x,y) \leq f(x_0,y_0)$$

(or $f(x,y) \geq f(x_0,y_0)$) for all points (x,y) in the domain of f, then f has an absolute maximum (or absolute minimum) at (x_0,y_0).

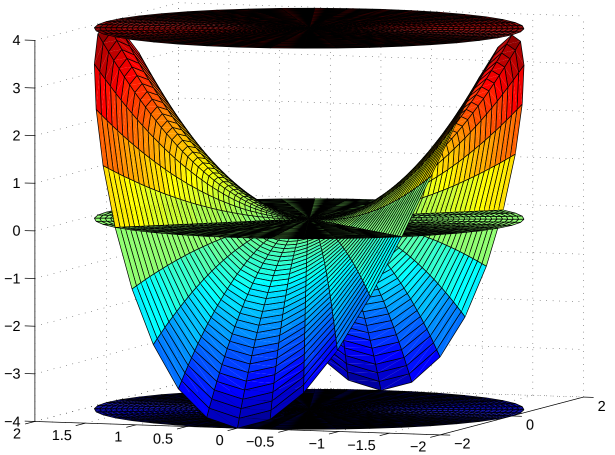
Absolute Maximum and Absolute Minimum

Definition (Absolute Maximum and Absolute Minimum on a set)

1. A number m is the absolute minimum of f on the set $D \subset D_f$, if there exists a point in this set, such that the value attained at this point equals m and for every point $(x, y) \in D$ the inequality holds
$$f(x, y) \geq m$$
2. A number M is the absolute maximum of f on the set $D \subset D_f$, if there exists a point in this set, such that the value attained at this point equals M and for every point $(x, y) \in D$ the inequality holds
$$f(x, y) \leq M$$

Theorem

If f is continuous on a closed, bounded set $D \subset \mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.


4
3
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−4
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1
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0
−0.5
−1
−1.5
−2
−2
0
2

Theorem

To find the absolute maximum and minimum values of a continuous function f on a closed, bounded set D:

1. *Find the values of f at the critical points of f in D.*
2. *Find the extreme values of f on the boundary of D.*
3. *The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.*

Example

Find the absolute maximum and minimum values of the function

1. $f(x,y) = x^2 - xy + 2y^2 + 3x + 2y + 1$, on the closed triangular region bounded by $x = 0$, $y = 0$, $y = -x - 5$
2. $f(x,y) = x^2 - y^2$, $D = \{(x,y);\ x^2 + y^2 \leq 4\}$.