Lecture

Double integrals

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Definition (partition of a rectangle)

The partition of a rectangle $R = \{(x, y); a \le x \le b, c \le y \le d\}$ we call a set P consisting of the rectangles $R_1, R_2, \ldots, R_n$, with disjoint interiors, fulfilling the rectangle R.

Notation:

- $\Delta x_k, \Delta y_k$ — size of a rectangle R_k.
- $diam(P) = \max\{\sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}; 1 \le k \le n\}$ — diameter of a partition P.

Definition (Riemann Sum)

Let f be bounded on the rectangle R and let P be the partition of that rectangle, let

$$\Theta = \{(x_1^*, y_1^*), (x_2^*, y_2^*), \ldots, (x_n^*, y_n^*), \}$$

be the set of points (one point in each subrectangle).
The number

$$\sum_{k=1}^{n} f(x_k^*, y_k^*)(\Delta x_k)(\Delta y_k).$$

we call the Riemann Sum of f corresponding to P and Θ.

The sum means that for each subrectangle we evaluate f at the chosen point and multiply by the area of the subrectangle, then add the results for each subrectangle.

Definition

The double integral of f over the rectangle R is

$$\iint\limits_R f(x,y)dxdy = \lim_{diam(P)\to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*)(\Delta x_k)(\Delta y_k),$$

when the limit exists.

A function f is called integrable if the limit in the definition exists.

Fact

Any continuous function is integrable.

Theorem (linearity of integral)

Let f and g be integrable on R and let α, β be two real numbers. Then

$$\iint\limits_R (\alpha f(x,y) + \beta g(x,y))dP = \alpha \iint\limits_R f(x,y)dP + \beta \iint\limits_R g(x,y)dP.$$

Theorem (additivity of integral)

If the function f is integrable on R, then for any partition of that rectangle into two rectangles R_1, R_2 with disjoint interiors we have

$$\iint\limits_R f(x,y)dxdy = \iint\limits_{R_1} f(x,y)dxdy + \iint\limits_{R_2} f(x,y)dxdy.$$

Theorem (iterated integration)

If the function f is continuous on the rectangle $[a, b] \times [c, d]$, *then*

$$\iint_{[a,b]\times[c,d]} f(x, y)dxdy = \int_a^b \left[\int_c^d f(x, y)dy\right] dx = \int_c^d \left[\int_a^b f(x, y)dx\right] dy.$$

So we can evaluate double integrals over rectangles by converting to either iterated integral.

Remark

Instead of $\int\limits_a^b \left[\int\limits_c^d f(x,y)dy \right] dx$ and $\int\limits_c^d \left[\int\limits_a^b f(x,y)dx \right] dy$ we can also write $\int\limits_a^b dx \int\limits_c^d f(x,y)dy$ and $\int\limits_c^d dy \int\limits_a^b f(x,y)dx$.

Example

Calculate iterated integrals:

1. $\int_0^4 dx \int_2^3 (x - y^2) dy$,
2. $\int_{-1}^2 dy \int_0^3 (x + xy^2) dx$.

Example

Calculate double integrals:

1. $\iint_R x^2 y^2 dxdy, \ R = [0, 1] \times [-1, 1]$,
2. $\iint_R \sin(x + y) dxdy, \ R = \left[-\frac{\pi}{4}, \frac{\pi}{4}\right] \times \left[0, \frac{\pi}{4}\right]$.

Theorem

If the function f can be written as $f(x,y) = g(x)h(y)$, where the functions g and h are continuous on intervals $[a,b]$ and $[c,d]$ respectively, then

$$\iint\limits_{[a,b]\times[c,d]} f(x,y)dxdy = \left(\int\limits_a^b g(x)dx\right) \cdot \left(\int\limits_c^d h(y)dy\right).$$

Example

Represent the following integrals as products and sums of integrals

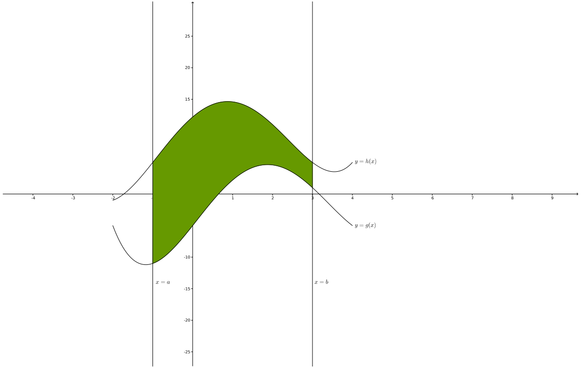
1. $\iint\limits_R e^{x+y}\,dxdy,\ \ R=[0,1]\times[-1,1],$
2. $\iint\limits_R \cos(x+y)\,dxdy,\ \ R=\left[-\frac{\pi}{4},\frac{\pi}{4}\right]\times\left[0,\frac{\pi}{4}\right].$

Definition

1. We call a region D the type I region, if

$$D = \{(x, y);\ a \le x \le b,\ g(x) \le y \le h(x)\},$$

where g and h are continuous on $[a, b]$.

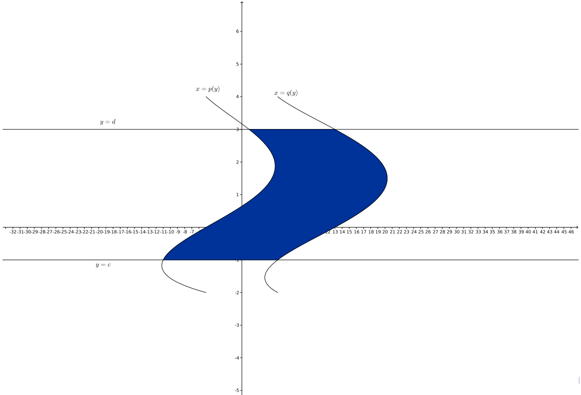


Definition

2. We call a region D the type II region, if

$$D = \{(x,y);\ p(y) \le x \le q(y),\ c \le y \le d\},$$

where p and q are continuous on $[c,d]$.



Example

Investigate if the given regions are of type I or II. Sketch these regions.

1. $y = 0,\ x = 1,\ y = x^2,$
2. $y = 2,\ x = 0,\ y = x^2,$
3. $y = -x^2 + 2,\ y = x^2,$
4. $y = 3x,\ y = x^2 - 2.$

Theorem

1. *If*

$$D = \{(x, y);\ a \le x \le b,\ g(x) \le y \le h(x)\}$$

is a type I region on which $f(x, y)$ *is continuous, then*

$$\iint_D f(x, y)dxdy = \int_a^b \left[\int_{g(x)}^{h(x)} f(x, y)dy \right] dx.$$

Theorem

1. *If*

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is a type II region on which $f(x,y)$ *is continuous, then*

$$\iint\limits_D f(x,y)dxdy = \int\limits_c^d \left[\int\limits_{p(y)}^{q(y)} f(x,y)dx \right] dy.$$

Example

Transform the double integral $\iint\limits_D f(x,y)dxdy$ into iterated integrals, if the region D is defined as:

1. $y = 1 + \sqrt{2x - x^2}, \; x = 0, \; x = 2, \; y = 0,$
2. $x = y^2, \; y = x - 2.$

Example

Calculate iterated integrals. Sketch the integration regions.

1. $\int\limits_0^3 dx \int\limits_x^{3x} (x - y)dy,$
2. $\int\limits_0^{\pi/2} dx \int\limits_0^{2x} \sin(x + y)dy.$

Example

Calculate double integrals over given regions:

1. $\iint\limits_D (x^2 - xy^2)dxdy, \ D = \{(x,y);\ y \geq x,\ y \leq 4x - x^2\},$
2. $\iint\limits_D x^2 ydxdy, \ D = \{(x,y);\ y \geq x^2,\ y \leq 3x - x^2\}.$

Fact

Let the region D be the union of two or more regions of type I or II, $D_1, D_2, \dots, D_n$ with disjoint interiors and let the function f be integrable on D. Then:

$$\iint\limits_D f(x,y)dxdy =$$
$$\iint\limits_{D_1} f(x,y)dxdy + \iint\limits_{D_2} f(x,y)dxdy + \dots + \iint\limits_{D_n} f(x,y)dxdy.$$

Example

Calculate integrals over the regions bounded by:

1. $\iint\limits_D xydxdy, \ D: \ y = x, y = \frac{1}{x}, \ y = 0, \ x = 4,$
2. $\iint\limits_D ydxdy, \ D: \ y = x^2, \ y = -x + 4, \ y = 0, \ x \geq 0.$

Example

Calculate integrals over the region $D: \ x^2 + y^2 \leq 3, \ x \geq 0, \ y \geq 0$:

1. $\iint\limits_D dxdy,$
2. $\iint\limits_D (x^2 + y^2)dxdy.$

Definition (the Jacobian)

Given the mapping $\tau(u,v) = (\varphi(u,v), \psi(u,v))$ we define the function:

$$J_\tau(u,v) = \det \begin{bmatrix} \frac{\partial\varphi(u,v)}{\partial u}(u,v) & \frac{\partial\varphi(u,v)}{\partial v}(u,v) \\ \frac{\partial\psi(u,v)}{\partial u}(u,v) & \frac{\partial\psi(u,v)}{\partial v}(u,v) \end{bmatrix}.$$

called the Jacobian.

Remark

We can also denote it as $\frac{\partial(\varphi,\psi)}{\partial(u,v)}$ or $\frac{D(\varphi,\psi)}{D(u,v)}$

Theorem (change of variables in double integrals)

Let

1. $\tau : \begin{cases} x = \varphi(u,v) \\ y = \psi(u,v) \end{cases}$ *be a transformation of the plane that is one to one from a region* Δ *in the* (u,v)*-plane to a region* D *in the* (x,y)*-plane,*
2. *functions* φ *and* ψ *have continuous partial derivatives on some open region containing* Δ*,*
3. *function* f *is continuous on* D*,*
4. *the Jacobian* J_τ *is never zero inside* D*.*

Then

$$\iint\limits_D f(x,y)dxdy = \iint\limits_\Delta f(\varphi(u,v),\psi(u,v))|J_\tau(u,v)|dudv.$$

Fact

Recall that the cartesian coordinates of the point given in the polar coordinates can be calculeted by

$$B : \begin{cases} x = r\cos\varphi \\ y = r\sin\varphi. \end{cases}$$

The Jacobian of the transformation B is r, thus

$$J_B(r,\varphi) = r.$$

Theorem

$$\iint\limits_{D} f(x,y)dxdy = \iint\limits_{\Delta} f(r\cos\varphi, r\sin\varphi)r\ drd\varphi.$$

Example

Introducing polar coordinates calculate:

1. $\iint\limits_D xy^2 dxdy, \ D: x^2+y^2 \le 4, \ x \ge 0,$
2. $\iint\limits_D \frac{\ln(x^2+y^2)}{x^2+y^2} dxdy, \ D: 1 \le x^2+y^2 \le 4, \ y \ge 0,$
3. $\iint\limits_D (x^2+y^2) dxdy, \ D: x^2+y^2-2x \le 0.$

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[1] $\int \sin^4 x dx = \frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$

Remark

If in polar coordinates Δ has the form

$$\Delta = \{(r, \varphi) : \ \alpha \le \varphi \le \beta, \ g(\varphi) \le r \le h(\varphi)\}$$

where g and h are continuous on $[\alpha, \beta] \subset [0, 2\pi]$, then

$$\iint\limits_{\Delta} f(r\cos\varphi, r\sin\varphi)r \ drd\varphi = \int\limits_{\alpha}^{\beta} d\varphi \int\limits_{g(\varphi)}^{h(\varphi)} f(r\cos\varphi, r\sin\varphi)rdr.$$

- **The area of a type I or type II region** $D \subset \mathbb{R}^2$ can be written in the form

$$|D| = \iint\limits_D dP.$$

- **The volume of the cylindrical solid** V between the surfaces $z = d(x, y)$ and $z = g(x, y)$ over $D \subset \mathbb{R}^2$ is given by:

$$V = \iint\limits_D [g(x, y) - d(x, y)]\, dP.$$

- **The area of the surface** S, given as a graph of function $z = f(x, y)$, where $(x, y) \in D$ is given by:

$$|S| = \iint\limits_D \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dP.$$

Example

Calculate:

1. the area of the region bounded by the curves: $y = e^x$, $y = \ln x$, $x + y = 1$, $x = 2$,
2. the volume of the cylindrical solid between the surfaces : $x^2 + y^2 = 1$, $x + y + z = 3$, $z = 0$,
3. the area of the surface $z = 8 - 4x - 2y$, where $x \geq 0$, $y \geq 0$, $z \geq 0$.