Lecture 01

Vector Spaces (Linear spaces)

M.W.

Mathematics Teaching and Distance Learning Centre
Gdańsk Technical University

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Definition (Vector space)

Let $\mathbb{F}$ be a field, whose elements are referred to as scalars. A vector space over $\mathbb{F}$ is non empty set V, whose elements are referred to as vectors with the following algebraic structure

- V is an additive group; that is; there is a fixed mapping $V \times V \to V$ denoted by

$$(x, y) \to x + y \tag{1}$$

  and satisfying the following axioms:

  1. $(x + y) + z = x + (y + z)$ (associative law)
  2. $x + y = y + x$ (commutative law)
  3. there exists a zero-vector 0; i.e. a vector such that $x + 0 = 0 + x = x$ for every $x \in V$
  4. To every vector x there is a vector $-x$ such that $x + (-x) = 0$

Definition (Vector space)

- There is a fixed mapping $\mathbb{F} \times V \to V$ denoted by

$$(\lambda, x) \to \lambda x \tag{2}$$

  and satisfying the axioms

  1. $(\lambda\mu)x = \lambda(\mu x)$ (associative law)
  2. $(\lambda + \mu)x = \lambda x + \mu x$
     $\lambda(x + y) = \lambda x + \lambda y$ (distributive laws)
  3. $1 \cdot x = x$ (1 unit element of $\mathbb{F}$)

A vector space over a field $\mathbb{F}$ is sometimes called an $\mathbb{F}$-space. A vector space over the real field is called a real vector space and a vector space over the complexed field is called a complex vector space.

Example

Let $\mathbb{F}$ be a field. The set $\mathbb{F}^{\mathbb{F}}$ of all functions from $\mathbb{F}$ to $\mathbb{F}$ is a vector space over $\mathbb{F}$, under the operations of ordinary addition and scalar multiplication of functions:

$$(f+g)(x) = f(x) + g(x)$$

and

$$(af)(x) = a(f(x))$$

Example

The set $\mathcal{M}_{m,n}(\mathbb{F})$ of all $m \times n$ matrices with entries in a field $\mathbb{F}$ is a vector space over $\mathbb{F}$, under the operation of matrix addition and scalar multiplication.

Example

The set $\mathbb{F}^n$ of all ordered n-tuples whose components lie in a field $\mathbb{F}$, is a vector space over $\mathbb{F}$, with addition and scalar multiplication defined component-wise:

$$(a_1, \ldots, a_n) + (b_1, \ldots, b_n) = (a_1 + b_1, \ldots, a_n + b_n)$$

and

$$c(a_1, \ldots, a_n) = (ca_1, \ldots, ca_n)$$

When convenient, we will also write the elements of $\mathbb{F}^n$ in the column form. When $\mathbb{F}$ is a finite field F_q with q elements, we write $V(n, q)$ for $\mathbb{F}_q^n$.

Example

Sequence spaces

- The set $Seq(\mathbb{F})$ of all infinite sequences with members from a filed $\mathbb{F}$ is a vector space under the component-wise operations

$$(s_n) + (t_n) = (s_n + t_n)$$

and

$$a(s_n) = (as_n)$$

- The set c_0 of all sequences of complex numbers that converge to 0
- The set ℓ^∞ of all bounded complex sequences
- Let p be a positive integer. The set ℓ^p of all complex sequences (s_n) for which

$$\sum_{n=1}^{\infty} |s_n|^p < \infty$$

under component-wise operations.

Exercise

Exercise

Check if $\mathbb{R}^2$ with canonical scalar multiplication and addition defined by the formula

1. $(x, y) \oplus (x', y') = (x + x', y + 3y')$
2. $(x, y) \oplus (x', y') = (x + x', y - y')$

is a vector space.

Exercise

Check if $\mathbb{R}^2$ with canonical addition and scalar multiplication defined by the formula

1. $r \odot (x, y) = (ry, rx)$
2. $r \odot (x, y) = (rx, r^2 y)$

is a vector space.

Definition

Let S be non-empty subset of a vector space V. A linear combination of vectors in S is an expression of the form

$$a_1 v_1 + a_2 v_2 + \ldots + a_n v_n \tag{3}$$

where $v_1, \ldots, v_n \in S$ and $a_1, \ldots, a_n \in \mathbb{F}$. The scalars are called coefficients of the linear combination. A linear combination is trivial if every coefficient a_i is zero. Otherwise, it is non-trivial.

Definition

A subspace of a vector space V is a subset S of V that is a vector space in its own right under the operations obtained by restricting the operations of V to S. We use the notation $S \subseteq V$ to indicate that S is a subspace of V and $S \subset V$ to indicate that S is a proper subspace of V, that is $S \subseteq V$ but $S \neq V$. The zero subspace of V is $\{0\}$.

Theorem

A non-empty subset S of a vector space V is a subspace of V if and only if S is closed under addition and scalar multiplication or, equivalently S is closed under linear combinations, that is

$$a, b \in \mathbb{F}, u, v \in S \Rightarrow au + bv \in S \tag{4}$$

Examples of subspaces

Example

Consider the vector space $V(n,2)$ of all binary n-tuples, that is, n-tuples of 0's and 1's. The weight $\mathcal{W}(v)$ of a vector $v \in V(n,2)$ is the number of non-zero coordinates in v. For instance, $\mathcal{W}(101010) = 3$. Let E_n be the set of all vectors in V of even weight. Then E_n is a subspace of $V(n,2)$.

Example

Any subspace of the vector space $V(n,q)$ is called a linear code. Linear codes are among the most important and most studied types of codes, because their structure allows for efficient encoding and decoding of information.

Exercise

Exercise

Check if the following subsets are subspaces of the vector space $\mathbb{R}^2$

1. $\{(x,-x);\ x \in \mathbb{R}\}$
2. $\{(x,x-1);\ x \in \mathbb{R}\}$
3. $\{(x,y);\ xy \geq 0\}$

Definition

Let S and T be subspaces of V. The sum $S + T$ is defined by

$$S + T = \{u + v; \quad u \in S, \ v \in T\} \tag{5}$$

The sum of subspaces S and T of V is a subspace of V.

Internal Direct Sums

Definition

A vector space V is the (internal) direct sum of a family $\mathcal{F} = \{S_i; \quad i \in I\}$ of subspaces of V, written

$$V = \bigoplus \mathcal{F} \text{ or } V = \bigoplus_{i \in I} \mathcal{F} \tag{6}$$

if the following holds

1. V is the sum (join) of the family $\mathcal{F}$:

$$V = \sum_{i \in I} S_i \tag{7}$$

2. For each $i \in I$

$$S_i \cap \left(\sum_{j \neq i} S_j \right) = \{0\} \tag{8}$$

In this case, each S_i is called a direct summand of V. If $\mathcal{F} = \{S_1, \ldots, S_n\}$ is a finite family, the direct sum is often written

$$V = S_1 \oplus \ldots \oplus S_n \tag{9}$$

Finally, if $V = S \oplus T$, then T is called a complement of S in V.

Definition

The subspace spanned (or subspace generated) by a non-empty set S of vectors in V is the set of all linear combinations of vectors from S:

$$\langle S \rangle = \operatorname{span}(S) = \{r_1 v_1 + \ldots + r_n v_n; \quad r_i \in \mathbb{F}, \ v_i \in S\} \tag{10}$$

When $S = \{v_1, \ldots, v_n\}$ is a finite set, we use the notation $\langle v_1, \ldots v_n \rangle$ or $\operatorname{span}(v_1, \ldots v_n)$. A set S of vectors in V is said to span V, or generate V, if $V = \operatorname{span}(S)$.

Definition

Let V be a vector space. A non-empty set S of vectors in V is linearly independent if fir any distinct vectors $s_1, \ldots, s_n$ in S

$$a_1 s_1 + \ldots + a_n s_n = 0 \Rightarrow a_i = 0 \text{ for all } i \tag{11}$$

In word, S linearly independent if the only linear combination of vectors from S that is equal to 0 is the trivial linear combination, all of whose coefficients are). If S is not linearly independent, it is said to be linearly dependent.

Definition

Let S be a non-empty set of vectors in V. To say that a non-zero vector $v \in V$ is an essentially unique linear combination of the vectors in S is to sat that, up to order of terms, there is one and only one way to express v as a linear combination

$$v = a_1 s_1 + \ldots + a_n s_n \tag{12}$$

where s_i's are distinct vectors in$S$ and the coefficients a_i are non-zero.

More explicitly $v \neq 0$ is an essentially unique linear combination of the vectors in S if $v \in \langle S \rangle$ and if whenever

$$v = a_1 s_1 + \ldots + a_n s_n \text{ and } v = b_1 t_1 + \ldots + b_m t_m$$

where s_i's are distinct the t_i's are distinct and all coefficients are non-zero then $n = m$ and after re-indexing of the $b_i t_i$'s if necessary, we have $a_i = b_i$ and $s_i = t_i$ for all $i = 1, \ldots, n$.

Theorem

Let $S \neq \{0\}$ be a non-empty set of vectors in V. The following are equivalent

1. *S is linearly independent.*
2. *Every non-zero vector $v \in \operatorname{span}(S)$ is an essentially unique linear combination of the vectors in S*
3. *No vector in S is a linear combination of the other vectors in S.*

Theorem

Lest S be a set of vectors in V. the following are equivalent:

1. *S is linearly independent and spans V*
2. *Every non-zero vector $v \in V$ is an essentially unique combination of vectors in S*
3. *S is minimal spanning linearly independent set, but any proper subset does not span V*
4. *S is a maximal linearly independent set,that is, S is linearly independent, but any proper superset of S is not linearly independent*

Definition

A set of vectors in V that satisfies any (and hence all) of above conditions is called a basis for V.

Theorem

A finite set $S = \{v_1, \ldots, v_n\}$ *of vectors in* V *is a basis for* V *if and only if*

$$V = \langle v_1 \rangle \oplus \ldots \oplus \langle v_n \rangle \tag{13}$$

Example

The ith standard vector in $\mathbb{F}^n$ is the vector e_i that has 0's in all coordinate positions except the ith, where it has a 1. Thus,

$$e_1 = (1, 0, \ldots, 0), \quad e_2 = (0, 1, 0, \ldots, 0), \ \ldots, \ e_n = (0, \ldots, 0, 1) \tag{14}$$

The set $\{e_1, \ldots, e_n\}$ is called the standard basis for $\mathbb{F}^n$.

Theorem

Let V be a non-zero vector space. Let I be a linearly independent set in V and let S be a spanning set in V containing I. Then there is a basis $\mathcal{B}$ for V which $I \subseteq \mathcal{B} \subseteq S$. In particular

1. *Any vector space, except the zero space $\{0\}$, has a basis.*
2. *Any linearly independent set in V is contained in a basis.*
3. *Any spanning set in V contains a basis.*

Example

Let S be an arbitrary set and consider the set $C(S)$ of all mappings $f\colon S \to \mathbb{F}$ such that $f(s) = 0$ for all but finitely many $s \in S$. Then if f and g are two such mappings, and λ is any scalar, the mappings $f + g$ and λf defined by

$$(f + g)(s) = f(s) + g(s)$$

and

$$(\lambda f)(s) = \lambda \cdot f(s)$$

are again contained in $C(S)$. Thus we make the set $C(S)$ into a vector space.
Now for each $a \in S$ denote by f_a the mapping given by

$$f_a(s) = \begin{cases} 1 & \text{if} \quad s = a \\ 0 & \text{if} \quad s \neq a \end{cases}$$

Then the vectors f_a are a basis of $C(S)$.

Theorem

Let V be vector space and assume that the vectors $v_1, \ldots, v_n$ are linearly independent and the vectors $s_1, \ldots, s_m$ span V. Then $n \leq m$.

Corollary

If V has a finite spanning set, then any two bases of V have the same size.

Theorem

If V is a vector space, then any two bases for V have the same cardinality.

Definition

A vector space V is finite-dimensional if it is the zero space $\{0\}$, or if it has a finite basis. All other vector spaces are infinite-dimensional. the dimension of the zero space is 0 and the dimension of any non-zero vector space V is the cardinality of any basis of V. If a vector space V has a basis of cardinality κ, we say that V is κ-dimensional and write $\dim(V) = \kappa$.

Theorem

Let V be a vector space

1. *If $\mathcal{B}$ is a basis of V and if $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ and $\mathcal{B}_1 \cap \mathcal{B}_2 = \emptyset$ then*

$$V = \langle \mathcal{B}_1 \rangle \oplus \langle \mathcal{B}_2 \rangle$$

2. *Let $V = S \oplus T$. If $\mathcal{B}_1$ is a basis for S and $\mathcal{B}_2$ is a basis for T, then $\mathcal{B}_1 \cap \mathcal{B}_2 = \emptyset$ and $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ is a basis for V.*

Theorem

Let S and T be subspaces of a vector space V. Then

$$\dim(S) + \dim(T) = \dim(T + S) + \dim(S \cap T) \tag{15}$$

In particular, if T is any complement of S in V, then

$$\dim(S) + \dim(T) = \dim(V) \tag{16}$$

that is,

$$\dim(S \oplus T) = \dim(S) + \dim(T) \tag{17}$$

Definition

Let V be a vector space of dimension n, An ordered basis for V is and ordered n-tuple $(v_1, \ldots, v_n)$ of vectors for which the set $\{v_1, \ldots, v_n\}$ is a basis for V

If $\mathcal{B} = (v_1, \ldots, v_n)$ is an ordered basis for V, then for each $v \in V$ there is a unique ordered n-tuple $(r_1, \ldots, r_n)$ of scalars for which

$$v = r_1 v_1 + \ldots + r_n v_n \tag{18}$$

Accordingly, we can define the coordinate map $\phi_{\mathcal{B}} \colon V \to \mathbb{F}^n$ by

$$\phi_{\mathcal{B}}(v) = [v]_{\mathcal{B}} = \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix} \tag{19}$$

where the column matrix $[v]_{\mathcal{B}}$ is known as the coordinate matrix of v with respect to the ordered basis $\mathcal{B}$. Clearly, knowing $[v]_{\mathcal{B}}$ is equivalently to knowing v (assuming knowledge of $\mathcal{B}$).

It is easy to see that the coordinate map $\phi_{\mathcal{B}}$ is bijective and preserves the vector space operations, that is

$$\phi_{\mathcal{B}}(r_1 v_1 + \ldots + r_n v_n) = r_1 \phi_{\mathcal{B}}(v_1) + \ldots + r_n \phi_{\mathcal{B}}(v_n)$$

or equivalently

$$[r_1 v_1 + \ldots + r_n v_n] = r_1 [v_1] + \ldots + r_n [v_n]$$

Functions from one vector space to another that preserve the vector space operations are called linear transformations.

Example

Given the basis $\mathcal{B} = (b_1, b_2) = \left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right)$ of the vector space $\mathbb{R}^2$. Find the coordinate vector $[v]_{\mathcal{B}}$ of the vector $v = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$. Now given the coordinate vector $[x]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$ find the vector $x \in \mathbb{R}^2$.

Exercises

1. Which of the following sets of vectors in $\mathbb{R}^4$ are linearly independent, (a generating set, a basis)?
   1. $(1,1,1,1),(1,0,0,0),(0,1,0,0),(0,0,0,1)$
   2. $(1,0,0,0),(2,0,0,0)$
   3. $(17,39,25,10),(13,12,99,4),(16,1,0,0)$
   4. $(1,\frac{1}{2},0,0),(0,0,1,1),(0,\frac{1}{2},\frac{1}{2},1),(\frac{1}{4},0,0,\frac{1}{4})$

   Extend the linearly independent sets to bases.
2. Are the vectors $x_1=(1,0,1)$; $x_2=(i,1,0)$; $x_3=(i,2,1+i)$ linearly independent in $\mathbb{C}^3$? Express $x=(1,2,3)$ and $y=(i,i,i)$ as linear combinations of x_1,x_2,x_3.
3. Let S be any set and consider the set of maps

$$f : S \to \mathbb{F}^n$$

   such that $f(x)=0$ for all but finitely many $x \in S$. Make this set into vector space (denoted by $C(S,\mathbb{F}^n)$). Construct a basis for this vector space.

4. Consider the set of polynomial functions $f\colon \mathbb{R} \to \mathbb{R}$,

$$f(x) = \sum_{i=0}^{n} \alpha_i x^i.$$

   Make this set into a vector space, and construct a natural basis.

5. Let (ξ^1, ξ^2, ξ^3) be an arbitrary vector in $\mathbb{F}^3$. Which of the following subsets are subspaces?
   1. all vectors with $\xi^1 = \xi^2 = \xi^3$
   2. all vectors with $\xi^3 = 0$
   3. all vectors with $\xi^1 = \xi^2 - \xi^3$
   4. all vectors with $\xi^1 = 1$

6. Find subspaces F_a, F_b, F_c, F_d generated by the sets of previous exercise, and construct bases for these subspaces.

7. Find complementary spaces for subspaces of previous problem and construct bases for these complementary spaces. Show that there exists more then one complementary space for each given subspace.
8. Show that
   1. $\mathbb{F}^3 = F_a + F_b$
   2. $\mathbb{F}^3 = F_b + F_c$
   3. $\mathbb{F}^3 = F_a + F_c$

   Find the intersections $F_a \cap F_b$, $F_b \cap F_c$, $F_a \cap F_c$ and decide in which cases the sums above are direct.

9. Let (x_1, x_2) be a basis of a 2-dimensional vector space. Show that the vectors

$$\widetilde{x}_1 = x_1 + x_2, \quad \widetilde{x}_2 = x_1 - x_2$$

again form a basis. Let (ξ^1, ξ^2) and $(\widetilde{\xi}^1, \widetilde{\xi}^2)$ be the components of a vector x relative to the bases (x_1, x_2) and $(\widetilde{x}_1, \widetilde{x}_2)$ respectively. Express the components $(\widetilde{\xi}^1, \widetilde{\xi}^2)$ in terms of the components (ξ^1, ξ^2).

10. Consider an n-dimensional complex vector space E. Since the multiplication with real coefficients in particular is defined in E, this space may also be considered as a real vector space. Let $(z_1, \ldots, z_n)$ be a basis of E. Show that the vectors $z_1, \ldots, z_n, iz_1, \ldots, iz_n$ form a basis of E if E considered as a real vector space.

11. In $\mathbb{F}^4$ consider the subspace T of all vectors $\xi^1, \xi^2, \xi^3, \xi^4)$ satisfying $\xi^1 + 2\xi^2 = \xi^3 + 2\xi^4$. Show that the vectors: $x_1 = (1, 0, 1, 0)$ and $x_2 = (0, 1, 0, 1)$ are linearly independent and lie in T; then extend this set of two vectors to a basis of T.
12. Let $\alpha_1, \alpha_2, \alpha_3$ be fixed real numbers. Show that all vectors $(\eta^1, \eta^2, \eta^3, \eta^4)$ in $\mathbb{R}^4$ obeying $\eta^4 = \alpha_1\eta_1 + \alpha_2\eta_2 + \alpha_3\eta_3$ form a subspace V. Show that V is generated by

$$x_1 = (1, 0, 0, \alpha_1),\ x_2 = (0, 1, 0, \alpha_2),\ x_3 = (0, 0, 1, \alpha_3).$$

Verify that x_1, x_2, x_3 form a basis of the subspace V.