Lecture 02

Linear transformations and linear functionals

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Definition

Let V and W be vector spaces over a field $\mathbb{F}$. A function $\tau\colon V \to W$ is a linear transformation if

$$\tau(ru + sv) = r\tau(u) + s\tau(v) \tag{1}$$

for all scalars $r, s \in \mathbb{F}$ and vectors $u, v \in V$. The set of all linear transformations from V to W is denoted by $\mathcal{L}(V, W)$

- A linear transformation from V to V is called a linear operator on V. The set of all linear operators on V is denoted by $\mathcal{L}(V)$. A linear operator on a real vector space is called real operator and a linear operator on a complex vector space is called a complex operator.
- A linear transformation from V to the base field $\mathbb{F}$ (thought of as a vector space over itself) is called a linear functional on V. The set of all linear functionals on V is denoted by V^* and called the dual space of V.

Definition

The following terms are also employed:

- homomorphism for linear transformation
- endomorphism for linear operator
- monomorphism (or embedding) for injective transformation
- epimorphism for surjective linear transformation
- isomorphism for bijective linear transformation
- automorphism for bijective linear operator

Examples

Example

1. The derivative $D\colon V \to V$ is a linear operator on the vector space V of all infinitely differentiable functions on $\mathbb{R}$.
2. The integral operator $\tau\colon \mathbb{F}[x] \to \mathbb{F}[x]$ defined

$$\tau f = \int_0^x f(t)dt$$

   is linear operator on $\mathbb{F}[x]$.
3. Let A be an $m \times n$ matrix over $\mathbb{F}$. The function $\tau_A\colon \mathbb{F}^n \to \mathbb{F}^m$ defined by $\tau_A v = Av$, where all vectors are written as column vectors, is linear transformation from $\mathbb{F}^n$ to $\mathbb{F}^m$.
4. The coordinate map $\phi\colon V \to \mathbb{F}^n$ of an n-dimensional vector space is linear transformation from V to $\mathbb{F}^n$.

Theorem

1. *The set* $\mathcal{L}(V,W)$ *is a vector space under ordinary addition of functions and scalar multiplication of function by elements of* $\mathbb{F}$
2. *If* $\sigma \in \mathcal{L}(U,V)$ *and* $\tau \in \mathcal{L}(V,W)$ *, then the composition* $\tau\sigma$ *is in* $\mathcal{L}(U,W)$
3. *If* $\tau \in \mathcal{L}(V,W)$ *is bijective then* $\tau^{-1} \in \mathcal{L}(W,V)$

Theorem

Let V and W be vector spaces and let $\mathcal{B} = \{v_i;\ i \in I\}$ be a basis for V. Then we can define a linear transformation $\tau \in \mathcal{L}(V, W)$ by specifying the values of τv_i arbitrarily for all $v_i \in \mathcal{B}$ and extending τ to V by linearity, that is,

$$\tau(a_1 v_1 + \ldots + a_n v_n) = a_1 \tau v_1 + \ldots + a_n \tau v_n \tag{2}$$

This process defines a unique linear transformation, that is, if $\tau, \sigma \in \mathcal{L}(V, W)$ satisfy $\tau v_i = \sigma v_i$ for all $v_i \in \mathcal{B}$ then $\tau = \sigma$.

Definition

Let $\tau \in \mathcal{L}(V, W)$. The subspace

$$\ker(\tau) = \{v \in V;\ \tau v = 0\} \tag{3}$$

is called the kernel of τ and the subspace

$$\mathrm{im}\,(\tau) = \{\tau v;\ v \in V\} \tag{4}$$

is called the image of τ. The dimension of $\ker(\tau)$ is called the nullity of τ and is denoted by $\mathrm{null}\,(\tau)$. The dimension of $\mathrm{im}\,(\tau)$ is called the rank of τ and is denoted by $\mathrm{rk}\,(\tau)$.

Theorem

Let $\tau \in \mathcal{L}(V, W)$. *Then*

- τ *is surjective if and only if* $im(\tau) = W$
- τ *is injective if and only if* $\ker(\tau) = \{0\}$

Definition

A bijective linear transformation $\tau \colon V \to W$ is called an isomorphism form V to W. When an isomorphism form V to W exists, we say that V and W are isomorphic and write $V \cong W$.

Example

Let $\dim(V) = n$. For any ordered basis $\mathcal{B}$ of V, the coordinate map $\phi_B \colon V \to \mathbb{F}^n$ that sends each vector $v \in V$ to its coordinate matrix $[V]_{\mathcal{B}} \in \mathbb{F}^n$ is an isomorphism. Hence any n-dimensional vector space over $\mathbb{F}$ is isomorphic to $\mathbb{F}^n$.

Theorem

Let $\tau \in \mathcal{L}(V, W)$ *be an isomorphism. Let* $S \subseteq V$*. Then*

1. S *spans* V *if and only if* τS *spans* W
2. S *is linearly independent in* V *if and only if* τS *is linearly independent in* W
3. S *is a basis for* V *if and only if* τS *is a basis for* W*.*

Theorem

A linear transformation $\tau \in \mathcal{L}(V, W)$ *is an isomorphism if and only if there is a basis* $\mathcal{B}$ *for* V *for which* $\tau\mathcal{B}$ *is a basis for* W*. In this case,* τ *maps any basis of* V *to a basis of* W*.*

Theorem

Let V *and* W *be vector spaces over* $\mathbb{F}$*. Then* $V \cong W$ *if and only if* $\dim(V) = \dim(W)$*.*

Theorem

1. *If A is a $m \times n$ matrix over $\mathbb{F}$. Denote as*

$$\tau_A(v) = Av,$$

   then $\tau_A \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$.

2. *If $\tau \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$ then $\tau = \tau_A$, where*

$$A = (\tau e_1 | \ldots | \tau e_n) \tag{5}$$

   The matrix A is called the matrix of τ.

Example

Example

Consider the linear transformation $\tau\colon \mathbb{F}^3 \to \mathbb{F}^3$ defined by

$$\tau(x, y, z) = (x - 2y, z, x + y + z)$$

Then we have, in column form,

$$\tau \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x - 2y \\ z \\ x + y + z \end{bmatrix} = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

and so the standard matrix of τ is

$$A = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Change of Basis Matrices

Theorem

Let $\mathcal{B} = (b_1, \ldots, b_n)$ and $\mathcal{C} = (c_1, \ldots, c_n)$ be ordered bases for a vector space V. Then the change of basis operator $\phi_{\mathcal{B},\mathcal{C}} = \phi_{\mathcal{C}}\phi_{\mathcal{B}}^{-1}$ is an automorphism of $\mathbb{F}^n$ whose standard matrix is

$$M_{\mathcal{B},\mathcal{C}} = ([b_1]_{\mathcal{C}} | \ldots | [b_n]_{\mathcal{C}}) \tag{6}$$

Hence

$$[v]_{\mathcal{C}} = M_{\mathcal{B},\mathcal{C}}[v]_{\mathcal{B}} \tag{7}$$

and $M_{\mathcal{C},\mathcal{B}} = M_{\mathcal{B},\mathcal{C}}^{-1}$.

The matrix of a Linear Transformation

Theorem

Let $\tau \in \mathcal{L}(V, W)$ and let $\mathcal{B} = (b_1, \ldots, b_n)$ and $\mathcal{C} = (c_1, \ldots, c_m)$ be ordered bases for V and W respectively. Then τ can be represented with respect to $\mathcal{B}$ and $\mathcal{C}$ as matrix multiplication, that is,

$$[\tau v]_{\mathcal{C}} = [\tau]_{\mathcal{B},\mathcal{C}}[v]_{\mathcal{B}} \tag{8}$$

where

$$[\tau]_{\mathcal{B},\mathcal{C}} = ([\tau b_1]_{\mathcal{C}} | \ldots [\tau b_n]_{\mathcal{C}}) \tag{9}$$

is called the matrix of τ with respect to the bases $\mathcal{B}$ and $\mathcal{C}$. When $V = W$ and $\mathcal{B} = \mathcal{C}$, we denote $[\tau]_{\mathcal{B},\mathcal{B}}$ by $[\tau]_{\mathcal{B}}$ and so

$$[\tau v]_{\mathcal{B}} = [\tau]_{\mathcal{B}}[v]_{\mathcal{B}} \tag{10}$$

Example

Let $D\colon \mathcal{P}_2 \to \mathcal{P}_2$ be the derivative operator, defined on the vector space of all polynomials of degree at most 2. Let $\mathcal{B} = \mathcal{C} = (1, x, x^2)$. Then

$$[D(1)]_{\mathcal{C}} = [0]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},\ [D(x)]_{\mathcal{C}} = [1]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ [D(x^2)]_{\mathcal{C}} = [2x]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

and so

$$[D]_{\mathcal{C}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Example

Example

Hence , for example, if $p(x) = 5 + x + 2x^2$, then

$$[Dp(x)]_{\mathcal{C}} = [D]_{\mathcal{B}}[p(x)]_{\mathcal{B}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}$$

and so $Dp(x) = 1 + 4x$.

Exercises

1. Using definition find the change of basis matrix from the base $\mathcal{B}$ to $\mathcal{B}'$ of the vector space $\mathbb{R}^2$, if
   - $\mathcal{B} = ([1,1],[3,2]), \quad \mathcal{B}' = ([3,4],[9,8])$
   - $\mathcal{B} = ([7,3],[9,4]), \quad \mathcal{B}' = ([1,0],[16,7])$
   - $\mathcal{B} = ([5,2],[4,1]), \quad \mathcal{B}' = ([9,3],[-1,2])$
2. Using properties of the change of basis matrix, find the change of basis matrix from the base $\mathcal{B}$ to $\mathcal{B}'$ of the vector space $\mathbb{R}^3$, if
   - $\mathcal{B} = ([1,2,3],[1,3,4],[1,5,7]), \quad \mathcal{B}' = ([2,3,4],[4,4,5],[6,3,4])$
   - $\mathcal{B} = ([5,2,4],[3,1,1],[5,1,2]), \quad \mathcal{B}' = ([5,3,6],[16,1,0],[5,2,4])$

Exercises

1. Having the following information, find the matrix $M_{\mathcal{BC}}(\varphi)$ of the linear transformation $\varphi\colon \mathbb{R}^3 \to \mathbb{R}^2$
   - $\varphi([x_1, x_2, x_3]) = [4x_1 + x_2 - 3x_3, 7x_1 + 2x_2 - 5x_3]$
     $\mathcal{B} = ([-2, 9, 0], [4, 0, 5], [0, 7, 2]), \quad \mathcal{C} = ([1, 4], [2, 7])$
   - $\varphi([x_1, x_2, x_3]) = [5x_1 + 3x_2 - 3x_3, 6x_1 + 4x_2 - 5x_3]$
     $\mathcal{B} = ([4, 4, 1], [5, 8, 2], [4, 5, 11]), \quad \mathcal{C} = ([5, 6], [4, 5])$
2. Linear transformation $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^3$ is given by
   - $[4, 5] \to [-1, 2, 5], \quad [5, 7] \to [-2, 1, 4]$
   - $[1, -2] \to [1, 3, 1], \quad [3, -5] \to [6, 10, 4]$

   Find the formula of $\varphi([x_1, x_2])$.