Lecture

Inner product

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1. Inner product

2. Hilbert spaces - examples

3. Applications

Definition

Let V be a vector space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. An inner product on V is a function $\langle , \rangle : V \times V \to \mathbb{F}$ with the following properties

1. (Positive definiteness) For all $v \in V$,
$$\langle v, v \rangle \geq 0 \text{ and } \langle v, v \rangle = 0 \Leftrightarrow v = 0 \tag{1}$$

2. For $\mathbb{F} = \mathbb{C}$: (Conjugacy symmetry)
$$\langle u, v \rangle = \overline{\langle v, u \rangle} \tag{2}$$
For $\mathbb{F} = \mathbb{R}$: (Symmetry)
$$\langle u, v \rangle = \langle v, u \rangle \tag{3}$$

3. (Linearity in the first coordinate) For all $u, v \in V$ and $r, s \in \mathbb{F}$
$$\langle ru + sv, w \rangle = r\langle u, w \rangle + s\langle v, w \rangle \tag{4}$$

Definition

A real (or complex) vector space V, together with an inner product, is called a real (or complexed) inner product space.

If $\mathbb{F} = \mathbb{R}$, then properties 2 i 3 imply that the inner product is linear in both coordinates , that is, the inner product is bilinear. However, if $\mathbb{F} = \mathbb{C}$, then

$$\langle w, ru + sv \rangle = \overline{\langle ru + sv, w \rangle} = \overline{r}\langle w, u \rangle + \overline{s}\langle w, v \rangle \tag{5}$$

This is referred to as conjugate linearity in the second coordinate. A complex inner product is linear in its first coordinate and conjugate linear in its second coordinate. This is often described by saying that complex inner product is sesquilinear. (Sesqui means "one an a half times").

Examples

Example

1. The vector space $\mathbb{R}^n$ is an inner product space under the standard inner product, defined by

$$\langle (r_1, \ldots, r_n), (s_1, \ldots, s_n) \rangle = r_1 s_1 + \ldots + r_n s_n \tag{6}$$

   The inner product space $\mathbb{R}^n$ is often called *n*-dimensional Euclidean space

2. The vector space $\mathbb{C}^n$ is an inner product space under standard inner product defined by

$$\langle (r_1, \ldots, r_n), (s_1, \ldots, s_n) \rangle = r_1 \overline{s}_1 + \ldots + r_n \overline{s}_n \tag{7}$$

   This inner product space is often called *n*-dimensional unitary space.

Example

3. The vector space $C[a, b]$ of all continuous complex-valued functions on the closed interval $[a, b]$ is a complex inner product space under the inner product

$$\langle f, g \rangle = \int_a^b f(x)\overline{g(x)}dx \tag{8}$$

Important example

Example

One of the most important inner product spaces is the vector space ℓ^2 of all real (or complexed) sequences (s_n) with the property that

$$\sum |s_n|^2 < \infty \tag{9}$$

under the inner product

$$\langle (s_n), (t_n) \rangle = \sum_{n=1}^{\infty} s_n \overline{t}_n \tag{10}$$

Such sequences are called square summable.

Norm

If V is an inner product space, the norm, or length of $v \in V$ is defined by

$$||v|| = \sqrt{\langle v, v \rangle} \tag{11}$$

A vector v is a unit vector if $||v|| = 1$.

Theorem

1. $||V|| \geq 0$ *and* $||v|| = 0$ *if and only if* $v = 0$
2. *For all* $r \in \mathbb{F}$ *and* $v \in V$,
$$||rv|| = |r| \; ||v||$$
3. *(The Cauchy-Schwartz inequality) For all* $v, u \in V$
$$|\langle u, v\rangle| \leq ||u|| \; ||v|| \tag{12}$$
with equality if and only if one of u *and* v *is a scalar multiple of the other.*
4. *(The triangle inequality) For all* $u, v \in V$
$$||u + v|| \leq ||u|| + ||v|| \tag{13}$$
with equality if and only if one of u *and* v *is a scalar multiple of the other.*

Theorem

5. *For all $u, v, x \in V$*
$$||u - v|| \leq ||u - x|| + ||x - v|| \tag{14}$$
6. *For all $u, v \in V$*
$$|\ ||u|| - ||v||\ | \leq ||u - v|| \tag{15}$$
7. *(The parallelogram law) For all $u, v \in V$*
$$||u + v||^2 + ||u - v||^2 = 2||u||^2 + 2||v||^2 \tag{16}$$

Any vector space V, together with function $||\cdot||\colon V \to \mathbb{R}$ that satisfies properties 1), 2) and 4) of the previous theorem is called a normed linear space and the function $||\cdot||$ is called a norm. Thus, any inner product space is a normed linear space, under norm given by (11)

Let V be an inner product space. The distance $d(u, v)$ between any two vectors u and v in V is

$$d(u, v) = ||u - v|| \tag{17}$$

Theorem

1. $d(u, v) \geq 0$ *and* $d(u, v) = 0$ *if and only if* $u = v$
2. *(Symmetry)*
$$d(u, v) = d(v, u) \tag{18}$$
3. *(The triangle inequality)*
$$d(u, v) \leq d(u, w) + d(w, v) \tag{19}$$

Any nonempty set V, together with a function $d\colon V \times V \to \mathbb{R}$ that satisfies the properties of the above theorem is called a metric space and the function d is called a metric on V. Thus, any inner product space is a metric space under the metric (17)

The presence of an inner product, and hence a metric, permits the definition of a topology on V, and in particular, convergence of infinite sequences. a sequence (v_n) of vectors in V converges to $v \in V$ if

$$\lim_{n\to\infty} ||v_n - v|| = 0 \tag{20}$$

Some of more important concepts related to convergence are closedness and closures, completeness and the continuity of linear operators and linear functionals.

Let M be a metric space with metric d

Definition

A sequence (x_n) in a metric space M is a Cauchy sequence if for any $\varepsilon > 0$, there exists an $N > 0$ for which

$$n, m > N \Rightarrow d(x_n, x_m) < \varepsilon \tag{21}$$

Definition

Let M be a metric space

1. M is said to be complete if every Cauchy sequence in M converges in M
2. A subspace S of M is complete if it is complete as a metric space. Thus S is complete if every Cauchy sequence (s_n) in S converges to an element in S.

Definition

Let $f\colon M \to M'$ be a function from the metric space (M, d) to the metric space (M', d'). we say that f is continuous at $x_0 \in M$ if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$d(x, x_0) < \delta \Rightarrow d'(f(x), f(x_0)) < \varepsilon \tag{22}$$

A function is continuous if it is continuous at every $x_0 \in M$.

Theorem

A function $f\colon M \to M'$ is continuous if and only if whenever (x_n) is a sequence in M that converges to $x_0 \in M$, then sequence $(f(x_n))$ converges to $f(x_0)$, in short

$$(x_n) \to x_0 \Rightarrow (f(x_n)) \to f(x_0) \tag{23}$$

Definition

An inner product space that is complete under the metric induced by the inner product is said to be a Hilbert space.

Example

The space ℓ^2 is a Hilbert space. The inner product is defined by

$$\langle x, y\rangle = \sum_{n=1}^{\infty} x_n \overline{y}_n$$

The metric induced by this inner product is

$$d(x, y) = ||x - y||_2 = \left(\sum_{n=1}^{\infty} |x_n - y_n|^2\right)^{1/2}$$

The space ℓ^2 is a prototype of all Hilbert spaces, introduced by David Hilbert in 1912, even before the axiomatic definition of Hilbert spaces was given by John von Neumann in 1927

In the finite-dimensional case, the situation is very straightforward:

- All subspaces are closed,
- all linear product spaces are complete
- and all linear operators and functionals are continuous.

However, in the infinite-dimensional case, things are not as simple.

Definition

Let V be an inner product space

1. Two vectors $u, v \in V$ are orthogonal, written $u \perp v$, if
$$\langle u, v \rangle = 0 \tag{24}$$
2. Two subspaces $X, Y \subseteq V$ are orthogonal, written $X \perp Y$, if $\langle X, Y \rangle = \{0\}$, that is, if $x \perp y$ for all $x \in X$ and $y \in Y$. We write $v \perp X$ in place of $\{v\} \perp X$.
3. The orthogonal complement of a subspace $X \subseteq V$ is the set
$$X^{\perp} = \{v \in V; \quad v \perp X\} \tag{25}$$

Definition

A nonempty set $\mathcal{O} = \{u_i; \quad i \in K\}$ of vectors in an inner product space is said to be an orthonormal set if $u_i \perp u_j$ for all $i \neq j \in K$. If, in addition, each vector u_i is a unit vector, then $\mathcal{O}$ is an orthonormal set. Thus, a set is orthonormal if

$$\langle u_i, u_j \rangle = \delta_{i,j} \tag{26}$$

for all $i, j \in K$, where $\delta_{i,j}$ is Kronecker delta function.

Theorem

Any orthonormal set of nonzero vectors in V is linearly independent.

Definition

A maximal orthonormal set in an inner product space V is called a Hilbert basis for V

Example

Let $V = \ell^2$ and let M be the set of all vectors of the form

$$e_i = (0, \ldots, 0, 1, 0, \ldots)$$

where e_i has 1 in the ith coordinate and 0's elsewhere. Clearly, M is an orthonormal set. Moreover, it is maximal. For if $v = (x_n) \in \ell^2$ has the property that $v \perp M$, then

$$x_i = \langle v, e_i \rangle = 0$$

for all i and so $v = 0$. Hence, no nonzero vector $v \notin M$ is orthogonal to M. This shows that M is a Hilbert basis for the inner product space ℓ^2.

Theorem

Let $\mathcal{O} = \{u_1, \ldots, u_k\}$ be an orthonormal subset of an inner product space V and let $S = \langle \mathcal{O} \rangle$. The Fourier expansion with respect to $\mathcal{O}$ of a vector $v \in V$ is

$$\widehat{v} = \langle v, u_1 \rangle u_1 + \ldots + \langle v, u_k \rangle u_k \tag{27}$$

Each coefficient $\langle v, u_i \rangle$ is called Fourier coefficient of v with respect to $\mathcal{O}$. The vector $\widehat{v}$ can be characterized as follows:

1. *$\widehat{v}$ is the unique vector $s \in S$ for which $(v - s) \perp S$*
2. *$\widehat{v}$ is the best approximation to v from within S, that is, $\widehat{v}$ is the unique vector $s \in S$ that is closest to v, in the sense that*

$$||v - \widehat{v}|| < ||v - s|| \tag{28}$$

   for all $s \in S \setminus \{\widehat{v}\}$

3. *Bessel's inequality holds for all $v \in V$, that is*

$$||\widehat{v}|| \leq ||v|| \tag{29}$$

Problems

1. Show, that in any Euclidean space equality $||u|| = ||v||$ holds if and only if $u + v \perp u - v$
2. In vector space $\mathbb{R}^2$ define inner product, such that the following conditions hold $[1, 0] \perp [1, 1]$, $[0, 1] \perp [2, 1]$ i $||[1, 0]|| = 2$
3. Check if the given vectors form an orthonormal basis of Euclidean space $\mathbb{R}^3$: $v_1 = \frac{1}{5}[4, 3, 0], v_1 = \frac{1}{5}[3, -4, 0], v_3 = [0, 0, 1]$
4. Applying Gram-Schmidt orthogonalization process, construct orthonormal basis of the given subspace of the Euclidean space $\mathbb{R}^4$: $span([1, 1, 1, 1], [3, 3, 1, 1], [7, 5, 3, 1])$

Let V and W be finite-dimensional inner product spaces over $\mathbb{F}$ and let $\tau \in \mathcal{L}(V, W)$. Then there is a unique function $\tau^*\colon W \to V$, defined by the condition

$$\langle \tau v, w \rangle = \langle v, \tau^* w \rangle \tag{30}$$

for all $v \in V$ and $w \in W$. This function is in (W, V) and is called the adjoint of τ.

Theorem

Let V and W be finite-dimensional inner product spaces. For every $\sigma, \tau \in \mathcal{L}(V, W)$ and $r \in \mathbb{F}$

1. $(\sigma + \tau)^* = \sigma^* + \tau^*$
2. $(r\tau)^* = \overline{r}\tau^*$
3. $\tau^{**} = \tau$ *and so*
$$\langle \tau^* v, w \rangle = \langle v, \tau w \rangle$$
4. *If τ is invertible, then* $(\tau^{-1})^* = (\tau^*)^{-1}$
5. *If $V = W$ and $p[x] \in \mathbb{R}[x]$, then* $p(\tau)^* = p(\tau^*)$

Theorem

Moreover, if $\tau \in \mathcal{L}(V)$ and S is a subspace of V, then

7. *S is τ-invariant if and only if $S^{\perp}$ is τ^*-invariant.*
8. *$(S, S^{\perp})$ reduces τ if and only if S is both τ-invariant and τ^*-invariant, in which case*

$$(\tau|_S)^* = (\tau^*)|_S$$

Definition

Let V be inner product space

1. $\tau \in \mathcal{L}(V)$ is self-adjoint (also called Hermitian in the complex case and symmetric in the real case) if
$$\tau^* = \tau \tag{31}$$
2. $\tau \in \mathcal{L}(V)$ is skew self-adjoint (also called skew-Hermitian in the complex case and skew-symmetric in the real case) if
$$\tau^* = -\tau \tag{32}$$
3. $\tau \in \mathcal{L}(V)$ is unitary in the complex case and orthogonal in the real case if τ is invertible and
$$\tau^* = \tau^{-1} \tag{33}$$

Definition

Let $\tau\colon H_1 \to H_2$ be a linear transformation from H_1 to H_2. Then τ is said to be bounded if

$$\sup_{x\neq 0} \frac{||\tau x||}{||x||} < \infty \tag{34}$$

If the supremum on the left is finite, we denote it by $||\tau||$ and call it the norm of τ.

Locally integrable functions

Definition

A function f defined on $\mathbb{R}$ is called locally integrable, if the integral

$$\int\limits_a^b f$$

exists for $-\infty < a < b < \infty$

Theorem

Let f, g be locally integrable functions. If g is bounded on $[a, b]$ for every $-\infty < a < b < \infty$, then the product $f \cdot g$ is a locally integrable function.

A complex-valued function f is integrable if and only if its real part Ref and its imaginary part Imf are integrable. Moreover, if f is integrable, then

$$\int f = \int Ref + i \int Imf \tag{35}$$

Definition

A complex-valued function $f : \mathbb{R} \to \mathbb{C}$ is called locally integrable if its real part and imaginary part are locally integrable.

Definition

The set of all functions $f \colon \mathbb{R} \to \mathbb{R}$ satisfying $\int\limits_{\mathbb{R}} f < \infty$ we denote $L^1(\mathbb{R})$

Definition

Functional

$$|| \cdot || \colon L^1(\mathbb{R}) \to \mathbb{R}$$

defined as

$$||f|| = \int |f|$$

is a norm of $L^1(\mathbb{R})$

Theorem

$L^1(\mathbb{R})$ is complete.

Definition

For a real $p > 1$ by $L^p(\mathbb{R})$ we denote the space of all complex-valued locally integrable functions f such that

$$|f|^p \in L^1(\mathbb{R})$$

Moreover

$$||f||_p = \left(\int |f|^p\right)^{1/p}$$

is a norm.

Theorem

1. *$L^p(\mathbb{R})$ is a vector space.*
2. *$L^p(\mathbb{R})$ is complete for every $1 \leq p < \infty$*

Example

Example

- The space $L^2(\mathbb{R})$ with an inner product defined as

$$\int_{-\infty}^{\infty} f(x)\overline{g(x)}dx$$

  is an inner product space.
- The spaces $L^2(\mathbb{R})$ and $L^2([a,b])$ are Hilbert spaces.

Definition

A sequence (x_n) of vectors in an inner product space E is called weakly convergent to a vector x in E if for every $y \in E$

$$\langle x_n, y \rangle \to \langle x, y \rangle, \quad n \to \infty$$

Notation

$$x_n \xrightarrow{w} x$$

Theorem

A strongly convergent sequence is weakly convergent (to the same limit)

$$x_n \to x \Rightarrow x_n \xrightarrow{w} x$$

Theorem

Weakly convergent sequences in a Hilbert space are bounded.

Example

Let

$$\varphi_n(x) = \frac{e^{inx}}{\sqrt{2\pi}}, \quad n \in \mathbb{Z}$$

The set $\{\varphi_n;\ n \in \mathbb{Z}\}$ is an orthonormal set in $L^2([-\pi, \pi])$

For $m \neq n$ we have

$$\langle \varphi_m, \varphi_n \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(m-n)x} dx = \frac{e^{\pi i(m-n)} - e^{-\pi i(m-n)}}{2\pi i(m-n)} = 0$$

On the other hand

$$\langle \varphi_n, \varphi_n \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(n-n)x} dx = 1$$

Thus $\langle \varphi_m, \varphi_n \rangle = \delta_{mn}$

Definition

An orthonormal sequence (x_n) in an inner product space E is said to be complete if for every $x \in E$ we have

$$x = \sum_{n=1}^{\infty} \langle x, x_n \rangle x_n \tag{36}$$

Example

Let $H = L^2([-\pi, \pi])$ and let $x_n(t) = \frac{1}{\sqrt{\pi}} \sin nt$ for $n = 1, 2, \ldots$. The sequence (x_n) is an orthonormal set in H. The sequence however is not complete. Let us take $x(t) = \cos(t)$ then we have

$$\sum_{n=1}^{\infty} \langle x, x_n \rangle x_n(t) = \sum_{n=1}^{\infty} \left[\frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} \cos t \sin ntdt \right] \frac{\sin nt}{\sqrt{\pi}} = \sum_{n=1}^{\infty} 0 \cdot \sin nt = 0 =$$

$$\neq \cos t$$

Theorem

An orthonormal sequence (x_n) in a Hilbert space H is complete if and only if for all $n \in \mathbb{N}$ $\langle x, x_n \rangle = 0$ implies $x = 0$

Example

The orthonormal system

$$\varphi_n(x) = \frac{e^{inx}}{\sqrt{2\pi}}, \quad n = 0, \pm 1, \pm 2, \ldots$$

is complete in $L^2([-\pi, \pi])$

Example

The sequence of functions

$$\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}}, \dots$$

is a complete orthonormal system in $L^2([-\pi, \pi])$.
The orthogonality follows from the following identities by simple integration:

- $2\cos nx \cos mx = \cos(n+m)x + \cos n - m)x$
- $2\sin nx \sin mx = \cos(n-m)x - \cos(n+m)x$
- $2\cos nx \sin mx = \sin(n+m)x - \sin(n-m)x$

Since $\int\limits_{-\pi}^{\pi} \cos^2 x dx = \int\limits_{-\pi}^{\pi} \sin^2 x dx = \pi$ the sequence is also orthonormal.
Completeness follows from completeness of previous example in view of the following identities $e^0 = 1$ and $e^{inx} = (\cos nx + i \sin nx)$

Definition

A Hilbert space is called separable if it contains a complete orthonormal sequence.

Finite dimensional Hilbert spaces are considered separable.

Example

Space $L^2([-\pi, \pi])$ is separable.

Example

Space ℓ^2 is separable.

Theorem

Every separable Hilbert space contains a countable dense subset.

Theorem

Every orthogonal set in a separable Hilbert space is countable.

In applied mathematics equations can be often written as operator equations of the form

$$Tx = x \tag{37}$$

where T is an operator in a Hilbert space and x unknown.