Wówczas otrzymamy

$$\int \frac{dx}{\sin^2 x \cos^2 x} = \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} dx = \int \frac{dx}{\cos^2 x} + \int \frac{dx}{\sin^2 x} = \operatorname{tg} x - \operatorname{ctg} x + C.$$

18. Stosujemy podstawienie uniwersalne

$$\operatorname{tg} \frac{x}{2} = t.$$

Zgodnie z punktem (iv) mamy

$$\int \frac{2 - \sin x}{2 + \cos x} dx = \int \frac{2 - \frac{2t}{1+t^2}}{2 + \frac{1-t^2}{1+t^2}} \frac{2}{1+t^2} dt = 2 \int \frac{2 + 2t^2 - 2t}{2 + 2t^2 + 1 - t^2} \frac{1}{1+t^2} dt$$

$$= 4 \int \frac{t^2 - t + 1}{(t^2+3)(t^2+1)} dt.$$

Funkcję podcałkową rozkładamy na ułamki proste

$$\frac{t^2 - t + 1}{(t^2+3)(t^2+1)} \equiv \frac{A + Bt}{t^2+3} + \frac{C + Dt}{t^2+1},$$

a więc

$$t^2 - t + 1 \equiv (A + Bt)(t^2+1) + (C + Dt)(t^2+3).$$

Porównując współczynniki przy równych potęgach t, mamy

$$\begin{array}{c|l} t^0 & 1 = A + 3C, \\ t & -1 = B + 3D, \\ t^2 & 1 = A + C, \\ t^3 & 0 = B + D. \end{array}$$

Stąd

$$A = 1, \quad B = \frac{1}{2}, \quad C = 0, \quad D = -\frac{1}{2}$$

jest rozwiązaniem tego układu.

Z powyższego wynika więc

$$\int \frac{2 - \sin x}{2 + \cos x} dx = 4 \int \frac{1 + \frac{1}{2}t}{t^2+3} dt + 4 \int \frac{-\frac{1}{2}t}{t^2+1} dt$$

$$= 4 \int \frac{dt}{t^2+3} + 2 \int \frac{t}{t^2+3} dt - 2 \int \frac{t}{t^2+1} dt$$

$$= \frac{4}{3} \int \frac{dt}{1 + \left(\frac{t}{\sqrt{3}}\right)^2} + \ln(t^2+3) - \ln(t^2+1),$$

gdyż w dwóch ostatnich całkach, w liczniku mamy pochodną mianownika.

Podstawmy

$$\frac{t}{\sqrt{3}} = u \quad \Longrightarrow \quad \frac{dt}{\sqrt{3}} = du.$$

Stosując twierdzenie o całkowaniu przez podstawienie, otrzymamy ostatecznie

$$\int \frac{2 - \sin x}{2 + \cos x} dx = \frac{4\sqrt{3}}{3} \int \frac{du}{1+u^2} + \ln \frac{t^2+3}{t^2+1} = \frac{4\sqrt{3}}{3} \operatorname{arctg} u + \ln \frac{t^2+3}{t^2+1} + C$$

$$= \frac{4\sqrt{3}}{3} \operatorname{arctg} \frac{t}{\sqrt{3}} + \ln \frac{t^2+3}{t^2+1} + C$$

$$= \frac{4\sqrt{3}}{3} \operatorname{arctg} \left(\frac{1}{\sqrt{3}} \operatorname{tg} \frac{x}{2} \right) + \ln \frac{\operatorname{tg}^2 \frac{x}{2} + 3}{\operatorname{tg}^2 \frac{x}{2} + 1} + C.$$

Obliczyć całki nieoznaczone:

1. $\int (3x^5 - x^2 + 1) dx$

2. $\int \frac{2x^2 - 3x + 1}{\sqrt{x}} dx$

3. $\int x^4 \sqrt[3]{x} dx$

4. $\int \frac{x^{-\frac{3}{4}} x^{-\frac{1}{4}}}{\sqrt[4]{x^5}} dx$

5. $\int \left(\frac{2}{x} - \frac{4}{x^2} - 5 \sin kx + \frac{1}{1+x^2} \right) dx, \quad k \neq 0$

6. $\int (2 + e^x)^2 dx$

7. $\int \left(e^x + e^{-x}\right)^2 dx$

8. $\int \left(\sin \frac{x}{2} - \cos \frac{x}{2} \right)^2 dx$

9. $\int (\operatorname{tg}^2 x + 1) dx$

10. $\int \left(\frac{1}{x^5} - \frac{5}{\sqrt{9 - 9x^2}} + \frac{1}{4 + 4x^2} \right) dx$

11. $\int \frac{(x^2 - 1)^3}{x} dx$

12. $\int \frac{\cos 2x}{\cos x - \sin x} dx$

13. $\int (\operatorname{tg} x + \operatorname{ctg} x)^2 dx$

14. $\int (\sqrt{x} + 1)(x - \sqrt{x} + 1) dx$

15. $\int \frac{x^2 + 7x + 12}{x + 4} dx$

16. $\int (2x - 1)^3 dx$

17. $\int \frac{1}{(2x - 1)^3} dx$

18. $\int \frac{x - 1}{x + 1} dx$

19. $\int \frac{x^2+4}{x^2+1}dx$

20. $\int \frac{3\sin^3\varphi+1}{\sin^2\varphi}d\varphi$

Stosując wzór na całkowanie przez części, obliczyć całki nieoznaczone:

21. $\int x\cos x dx$

22. $\int x\sin 2x dx$

23. $\int x\sin(2x+3)dx$

24. $\int xe^{-3x}dx$

25. $\int (1-2x)e^{-x}dx$

26. $\int x3^x dx$

27. $\int xe^{\frac{x}{2}}dx$

28. $\int x^2e^{-x}dx$

29. $\int x^9\ln x\,dx$

30. $\int \frac{\ln^2 x}{x^5}dx$

31. $\int x^2\cos x dx$

32. $\int \ln 3x dx$

33. $\int \ln(x-1)dx$

34. $\int x^2\ln x dx$

35. $\int \sqrt[3]{x}\ln x dx$

36. $\int \sqrt{x}\ln x dx$

37. $\int \frac{\ln x}{\sqrt[3]{x}}dx$

38. $\int \frac{\ln x}{\sqrt[4]{x}}dx$

39. $\int \ln^2 x dx$

40. $\int \frac{\ln\cos x}{\cos^2 x}dx$

41. $\int \frac{x}{\cos^2 x}dx$

42. $\int \frac{3x+5}{\sin^2 x}dx$

43. $\int x\sin^2 x dx$

44. $\int \frac{x^2\sin x}{\cos^3 x}dx$

45. $\int e^x\sin x dx$

46. $\int e^{-x}\cos x dx$

47. $\int e^{2x}\cos 3x dx$

48. $\int \sin(\ln x)dx$

49. $\int \arccos x dx$

50. $\int \operatorname{arctg}\frac{x}{2}dx$

51. $\int \operatorname{arcctg}3x dx$

52. $\int \frac{\arcsin x}{\sqrt{1+x}}dx$

Stosując wzór na całkowanie przez podstawienie, obliczyć całki nieoznaczone:

53. $\int x\sqrt{2-3x^2}dx$

54. $\int \frac{x}{\sqrt{1-x^2}}dx$

55. $\int x^2\sqrt{x^3+9}dx$

56. $\int xe^{1-x^2}dx$

57. $\int x\cos x^2 dx$

58. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx$

59. $\int \frac{e^{-\frac{2}{x}}}{x^2}dx$

60. $\int \frac{1}{(\arccos x)^3\sqrt{1-x^2}}dx$

61. $\int \frac{4-x}{2+\sqrt{x}}dx$

62. $\int \sqrt{1+4\sin x}\cos x dx$

63. $\int \frac{\sin x}{1+3\cos x}dx$

64. $\int \cos^3 x dx$

65. $\int \frac{\ln^3 x}{x}dx$

66. $\int \frac{\ln\sqrt{x}}{2\sqrt{x}}dx$

67. $\int \frac{1}{x(1+\ln x)}dx$

68. $\int \frac{\ln x}{x(\ln x+1)^2}dx$

69. $\int e^{\cos x}\sin x dx$

70. $\int \frac{1}{\cos^2 x\sqrt{1+\operatorname{tg}x}}dx$

71. $\int \frac{\sin x}{\sqrt{1+2\cos x}}dx$

72. $\int \frac{\cos(\ln x)}{x}dx$

73. $\int \frac{e^{\operatorname{arctg}2x}}{1+4x^2}dx$

74. $\int e^x\sqrt{3+4e^x}dx$

75. $\int \frac{1}{e^x+e^{-x}}dx$

76. $\int \frac{e^{2x}}{\sqrt[4]{1+e^x}}dx$

77. $\int \frac{x^3-1}{(x^4-4x)^{\frac{2}{3}}}dx$

78. $\int \frac{x}{(x^2+3)^6}dx$

79. $\int \frac{x}{x^4+1}dx$

80. $\int \frac{x^2}{(x^3+5)^4}dx$

81. $\int \frac{x^7}{(1+x^2)^5}dx$

82. $\int \frac{x^4}{x^{10}+3}dx$

83. $\int \frac{x}{(1+x)^{10}}dx$

84. $\int \frac{y^2}{\sqrt{1-y^6}}dy$

Stosując wzory na całkowanie przez części i podstawienie, obliczyć całki:

85. $\int x^3 e^{x^2} dx$

86. $\int \frac{e^{3x}}{e^{2x}+1} dx$

87. $\int \frac{x^2 \operatorname{arctg} x}{1+x^2} dx$

88. $\int \frac{x \arcsin x}{\sqrt{1-x^2}} dx$

89. $\int x^3 \arcsin \frac{1}{x} dx$

90. $\int (\arccos x)^2 dx$

91. $\int \operatorname{arctg} \sqrt{x} dx$

92. $\int \frac{\arcsin x}{x^2} dx$

93. $\int x^2 \arcsin x dx$

94. $\int x \operatorname{arctg} x^2 dx$

95. $\int e^{-x} \operatorname{arctg} e^x dx$

96. $\int \frac{\ln(\ln x)}{x} dx$

Obliczyć całki nieoznaczone funkcji wymiernych:

97. $\int \frac{1}{3x+5} dx$

98. $\int \frac{1}{(2x-3)^9} dx$

99. $\int \frac{1}{x^2+9} dx$

100. $\int \frac{3x+7}{x^2+9} dx$

101. $\int \frac{1}{x^2+2x+2} dx$

102. $\int \frac{1}{x^2-6x+9} dx$

103. $\int \frac{1}{x^2-5x+6} dx$

104. $\int \frac{1}{2x^2+3x+1} dx$

105. $\int \frac{1}{x^2-x} dx$

106. $\int \frac{1}{x^2-4} dx$

107. $\int \frac{1}{(x+2)(x-1)} dx$

108. $\int \frac{1}{(x+3)(x^2-3x+2)} dx$

109. $\int \frac{3x-4}{x^2-x-6} dx$

110. $\int \frac{x+1}{x^2-x+1} dx$

111. $\int \frac{x}{(x+1)(x^2+x+1)} dx$

112. $\int \frac{1}{x^4-1} dx$

113. $\int \frac{(1+x)^2}{x(1+x^2)} dx$

114. $\int \frac{2x+1}{(x-1)(x^2+2)} dx$

115. $\int \frac{1}{(x^2+3)(x^2+4)} dx$

116. $\int \frac{x^2+3}{(x+2)^2(x-2)} dx$

117. $\int \frac{3x^2+8}{x^3+4x^2+4x} dx$

118. $\int \frac{x^3+4x^2+6}{(x+1)^2(x^2+2)} dx$

119. $\int \frac{3x+5}{x^5+2x^3+x} dx$

120. $\int \frac{1}{x^3+1} dx$

121. $\int \frac{x^3}{x^2+1} dx$

122. $\int \frac{x^3-2x^2-1}{x^2-1} dx$

123. $\int \frac{2x^5+6x^3+1}{x^4+3x^2} dx$

124. $\int \frac{27x^6}{3x^2+2} dx$

Obliczyć całki nieoznaczone funkcji trygonometrycznych:

125. $\int \sin 3x dx$

126. $\int 5 \cos \frac{x}{2} dx$

127. $\int \sin 5x \cos 3x dx$

128. $\int \sin 3x \sin 2x dx$

129. $\int \cos(2x-1) \cos(4x+3) dx$

130. $\int \cos^2 x dx$

131. $\int \operatorname{tg}^2 4x dx$

132. $\int \sqrt{\cos x} \sin x dx$

133. $\int \frac{\cos x}{\sin^4 x} dx$

134. $\int \frac{\sin^3 x}{1+\cos^2 x} dx$

135. $\int \sin^2 x \cos^3 x dx$

136. $\int \sin^3 x dx$

137. $\int \sin^4 x dx$

138. $\int \sin^2 x \cos^2 x dx$

139. $\int \operatorname{tg}^3 x dx$

140. $\int \cos^4 x \sin^2 x dx$

141. $\int \frac{1}{5-3\cos x} dx$

142. $\int \frac{1}{3+5\cos x} dx$

143. $\int \frac{1}{3+2\sin x} dx$

144. $\int \frac{1}{\sin x - 2\cos x + 3} dx$

145. $\int \frac{1}{(3+\sin x)\cos x} dx$

146. $\int \frac{1}{3\sin x - 4\cos x} dx$

147. $\int \frac{\cos^2 x}{1+\sin^2 x} dx$

148. $\int \frac{\sin x \cos x}{1+\sin^4 x} dx$

Obliczyć całki nieoznaczone funkcji niewymiernych:

149. $\int \frac{1}{x^2} \sqrt{\frac{1+x}{x}} dx$

150. $\int \frac{1}{x} \sqrt{\frac{1-x}{1+x}} dx$

151. $\int \frac{1}{(x-1)^2}\sqrt[3]{\frac{1+x}{1-x}}dx$

152. $\int \frac{1}{\sqrt[3]{x}+\sqrt{x}}dx$

153. $\int \frac{1}{(1+\sqrt[3]{x})\sqrt{x}}dx$

154. $\int \frac{3-\sqrt[4]{x^3}}{\sqrt{x}}dx$

155. $\int \frac{1}{\sqrt{x}(\sqrt[3]{x}+1)^2}dx$

156. $\int \frac{\sqrt{x}}{x-(\sqrt[3]{x})^2}dx$

157. $\int \frac{1+\sqrt[3]{x+1}}{1+\sqrt{x+1}}dx$

158. $\int \frac{\sqrt[5]{x}}{\sqrt{3-2\sqrt[5]{x^3}}}dx$

159. $\int \frac{1}{\sqrt{7-x^2}}dx$

160. $\int \frac{1}{\sqrt{x^2-3}}dx$

161. $\int \frac{1}{\sqrt{1-(2x-1)^2}}dx$

162. $\int \frac{1}{\sqrt{x^2+2x-1}}dx$

163. $\int \frac{2x+3}{\sqrt{x^2+9}}dx$

164. $\int \frac{x-2}{\sqrt{16-x^2}}dx$

165. $\int \frac{2x+6}{\sqrt{x^2+4x+8}}dx$

166. $\int \sqrt{1-x^2}dx$

167. $\int \frac{x^2}{\sqrt{9+x^2}}dx$

168. $\int \sqrt{x^2-6x-7}dx$

Odpowiedzi

1. $\frac{1}{2}x^6-\frac{1}{3}x^3+x+C$

2. $\frac{4}{5}x^{\frac{5}{2}}-2x^{\frac{3}{2}}+2x^{\frac{1}{2}}+C$

3. $\frac{3}{16}x^{\frac{16}{3}}+C$

4. $-\frac{4}{5x\sqrt[4]{x}}+C$

5. $2\ln|x|+\frac{4}{x}+\frac{5}{k}\cos kx+\operatorname{arctg}x+C$

6. $4x+4e^x+\frac{1}{2}e^{2x}+C$

7. $\frac{1}{2}e^{2x}+2x-\frac{1}{2}e^{-2x}+C$

8. $x+\cos x+C$

9. $\operatorname{tg}x+C$

10. $-\frac{1}{4x^4}-\frac{5}{3}\arcsin x+\frac{1}{4}\operatorname{arctg}x+C$

11. $\frac{1}{6}x^6-\frac{3}{4}x^4+\frac{3}{2}x^2-\ln|x|+C$

12. $\sin x-\cos x+C$

13. $\operatorname{tg}x-\operatorname{ctg}x+C$

14. $\frac{2}{5}x^2\sqrt{x}+x+C$

15. $\frac{1}{2}x^2+3x+C$

16. $\frac{1}{8}(2x-1)^4+C$

17. $-\frac{1}{4}(2x-1)^{-2}+C$

18. $x-2\ln|x+1|+C$

19. $x+3\operatorname{arctg}x+C$

20. $-3\cos\varphi-\operatorname{ctg}\varphi+C$

21. $x\sin x+\cos x+C$

22. $-\frac{1}{2}x\cos 2x+\frac{1}{4}\sin 2x+C$

23. $-\frac{1}{2}x\cos(2x+3)+\frac{1}{4}\sin(2x+3)+C$

24. $-\frac{1}{9}(3x+1)e^{-3x}+C$

25. $(2x+1)e^{-x}+C$

26. $\frac{3^x}{\ln^2 3}(x\ln 3-1)+C$

27. $2e^{\frac{x}{2}}(x-2)+C$

28. $-e^{-x}(x^2+2x+2)+C$

29. $\frac{1}{10}x^{10}\left(\ln x-\frac{1}{10}\right)+C$

30. $-\frac{1}{32x^4}(8\ln^2 x+4\ln x+1)+C$

31. $(x^2-2)\sin x+2x\cos x+C$

32. $x(\ln 3x-1)+C$

33. $(x-1)\ln(x-1)-x+C$

34. $\frac{1}{3}x^3\left(\ln x-\frac{1}{3}\right)+C$

35. $\frac{3}{4}x^{\frac{4}{3}}(\ln x-\frac{3}{4})+C$

36. $\frac{2}{3}x^{\frac{3}{2}}(\ln x-\frac{2}{3})+C$

37. $\frac{3}{2}x^{\frac{2}{3}}(\ln x-\frac{3}{2})+C$

38. $\frac{4}{3}x^{\frac{3}{4}}(\ln x-\frac{4}{3})+C$

39. $x(\ln^2 x-2\ln x+2)+C$

40. $\operatorname{tg}x\ln\cos x+\operatorname{tg}x-x+C$

41. $x\operatorname{tg}x+\ln|\cos x|+C$

42. $-(3x+5)\operatorname{ctg}x+3\ln|\sin x|+C$

43. $\frac{1}{4}x^2-\frac{1}{4}x\sin 2x-\frac{1}{8}\cos 2x+C$

44. $\frac{x^2}{2\cos^2 x}-x\operatorname{tg}x-\ln|\cos x|+C$

45. $\frac{1}{2}e^x(\sin x-\cos x)+C$

46. $\frac{1}{2}e^{-x}(\sin x-\cos x)+C$

47. $\frac{1}{13}e^{2x}(2\cos 3x+3\sin 3x)+C$

48. $\frac{1}{2}x[\sin(\ln x)-\cos(\ln x)]+C$

49. $x\arccos x-\sqrt{1-x^2}+C;\ u=\arccos x,\ v'=1$

50. $x\operatorname{arctg}\frac{x}{2}-\ln(x^2+4)+C$

51. $x\operatorname{arcctg}3x+\frac{1}{6}\ln(9x^2+1)+C$

52. $2\sqrt{1+x}\arcsin x+4\sqrt{1-x}+C;\ u=\arcsin x,\ v'=\frac{1}{\sqrt{1+x}}$

53. $-\frac{1}{9}(2-3x^2)^{\frac{3}{2}}+C$

54. $-\sqrt{1-x^2}+C$

55. $\frac{2}{9}(x^3+9)^{\frac{3}{2}}+C$

56. $-\frac{1}{2}e^{1-x^2}+C$

57. $\frac{1}{2}\sin x^2+C$

58. $2e^{\sqrt{x}}+C$

59. $\frac{1}{2}e^{-\frac{2}{x}}+C$

60. $\frac{1}{2(\arccos x)^2}+C$

61. $2x-\frac{2}{3}x\sqrt{x}+C;\ \sqrt{x}=t$

62. $\frac{1}{6}(1+4\sin x)^{\frac{3}{2}}+C$

63. $-\frac{1}{3}\ln|1+3\cos x|+C$

64. $\sin x-\frac{1}{3}\sin^3 x+C$

65. $\frac{1}{4}\ln^4 x+C$

66. $\sqrt{x}(\ln\sqrt{x}-1)+C$

67. $\ln|1+\ln x|+C$

68. $\ln|\ln x+1|+\frac{1}{\ln x+1}+C$

69. $-e^{\cos x}+C$

70. $2\sqrt{1+\operatorname{tg}x}+C$

71. $-\sqrt{1+2\cos x}+C$

72. $\sin(\ln x)+C$

73. $\frac{1}{2}e^{\operatorname{arctg}2x}+C$

74. $\frac{1}{6}(3+4e^x)^{\frac{3}{2}}+C$

75. $\operatorname{arctg}e^x+C$

76. $\frac{4}{21}(3e^x-4)\sqrt[4]{(e^x+1)^3}+C$

77. $\frac{3}{4}(x^4-4x)^{\frac{1}{3}}+C;\ \ x^4-4x=t$

78. $-\frac{1}{10}(x^2+3)^{-5}+C$

79. $\frac{1}{2}\operatorname{arctg}x^2+C;\ \ x^2=t$

80. $-\frac{1}{9}(x^3+5)^{-3}+C$

81. $\frac{1}{2(1+x^2)}\left[-1+\frac{3}{2(1+x^2)}-\frac{1}{(1+x^2)^2}+\frac{1}{(1+x^2)^3}\right]+C;\ \ 1+x^2=t$

82. $\frac{1}{5\sqrt{3}}\operatorname{arctg}\frac{x^5}{\sqrt{3}}+C;\ \ x^5=t$

83. $\frac{1}{9(x+1)^9}-\frac{1}{8(x+1)^8}+C;\ \ 1+x=t$

84. $\frac{1}{3}\arcsin y^3+C;\ \ y^3=t$

85. $\frac{1}{2}e^{x^2}(x^2-1)+C$

86. $e^x-\operatorname{arctg}e^x+C;\ \ e^x=t$

87. $x\operatorname{arctg}x-\frac{1}{2}\ln(1+x^2)-\frac{1}{2}(\operatorname{arctg}x)^2+C;\ \ \operatorname{arctg}x=t$

88. $x-\sqrt{1-x^2}\arcsin x+C;\ \ \arcsin x=t$

89. $\frac{1}{4}\left[x^4\arcsin\frac{1}{x}+sgn(x)\frac{x^2+2}{3}\sqrt{x^2-1}\right]+C;\ u=\arcsin\frac{1}{x},v'=x^3$

90. $x(\arccos x)^2-2\sqrt{1-x^2}\arccos x-2x+C;\ \ \arccos x=t$

91. $x\operatorname{arctg}\sqrt{x}+\operatorname{arctg}\sqrt{x}-\sqrt{x}+C;\ \ u=\operatorname{arctg}\sqrt{x},\ v'=1$

92. $\ln\frac{1-\sqrt{1-x^2}}{|x|}-\frac{1}{x}\arcsin x+C$

93. $\frac{1}{3}x^3\arcsin x+\frac{1}{9}x^2\sqrt{1-x^2}+\frac{2}{9}\sqrt{1-x^2}+C$

94. $\frac{1}{2}x^2\operatorname{arctg}x^2-\frac{1}{4}\ln(1+x^4)+C;\ \ x^2=t$

95. $x-e^{-x}\operatorname{arctg}e^x-\frac{1}{2}\ln(1+e^{2x})+C;\ \ e^x=t$

96. $[\ln(\ln x)-1]\ln x+C;\ \ \ln x=t$

97. $\frac{1}{3}\ln|3x+5|+C$

98. $-\frac{1}{16}(2x-3)^{-8}+C$

99. $\frac{1}{3}\operatorname{arctg}\frac{x}{3}+C$

100. $\frac{3}{2}\ln(x^2+9)+\frac{7}{3}\operatorname{arctg}\frac{x}{3}+C$

101. $\operatorname{arctg}(x+1)+C$

102. $-\frac{1}{x-3}+C$

103. $\ln\left|\frac{x-3}{x-2}\right|+C$

104. $\ln\left|\frac{2x+1}{x+1}\right|+C$

105. $\ln\left|\frac{x-1}{x}\right|+C$

106. $\frac{1}{4}\ln\left|\frac{x-2}{x+2}\right|+C$

107. $\frac{1}{3}\ln\left|\frac{x-1}{x+2}\right|+C$

108. $\frac{1}{20}\ln\frac{|x+3|(x-2)^4}{|x-1|^5}+C$

109. $\ln|(x-3)(x+2)^2|+C$

110. $\frac{1}{2}\ln(x^2-x+1)+\sqrt{3}\operatorname{arctg}\frac{2x-1}{\sqrt{3}}+C$

111. $-\ln|x+1|+\frac{1}{2}\ln(x^2+x+1)+\frac{1}{\sqrt{3}}\operatorname{arctg}\frac{2x+1}{\sqrt{3}}+C$

112. $\frac{1}{4}\ln\left|\frac{x-1}{x+1}\right|-\frac{1}{2}\operatorname{arctg}x+C$

113. $2\operatorname{arctg}x+\ln|x|+C$

114. $\ln\frac{|x-1|}{\sqrt{x^2+2}}+\frac{1}{\sqrt{2}}\operatorname{arctg}\frac{x}{\sqrt{2}}+C$

115. $\frac{1}{\sqrt{3}}\operatorname{arctg}\frac{x}{\sqrt{3}}-\frac{1}{2}\operatorname{arctg}\frac{x}{2}+C$

116. $\frac{1}{16}\ln\left|(x+2)^9(x-2)^7\right|+\frac{7}{4(x+2)}+C$

117. $\ln\left[x^2|x+2|\right]+\frac{10}{x+2}+C$

118. $-\frac{3}{x+1}+\frac{1}{3}\ln\left[(2+x^2)|x+1|\right]-\frac{\sqrt{2}}{3}\operatorname{arctg}\frac{x}{\sqrt{2}}+C$

119. $5\ln\frac{|x|}{\sqrt{x^2+1}}+\frac{3x+5}{2(x^2+1)}+\frac{3}{2}\operatorname{arctg}x+C$

120. $\frac{1}{6}\ln\frac{(x+1)^2}{x^2-x+1}+\frac{1}{\sqrt{3}}\operatorname{arctg}\frac{2x-1}{\sqrt{3}}+C$

121. $\frac{1}{2}x^2-\frac{1}{2}\ln(x^2+1)+C$

122. $\frac{1}{2}x^2-2x+\ln\frac{(x+1)^2}{|x-1|}+C$

123. $x^2-\frac{1}{3x}-\frac{1}{3\sqrt{3}}\operatorname{arctg}\frac{x}{\sqrt{3}}+C$

124. $\frac{9}{5}x^5-2x^3+4x-\frac{4\sqrt{6}}{3}\operatorname{arctg}\sqrt{\frac{3}{2}}x+C$

125. $-\frac{1}{3}\cos 3x+C$

126. $10\sin\frac{x}{2}+C$

127. $-\frac{1}{4}\cos 2x-\frac{1}{16}\cos 8x+C$

128. $-\frac{1}{10}\sin 5x+\frac{1}{2}\sin x+C$

129. $\frac{1}{12}\sin(6x+2)+\frac{1}{4}\sin(2x+4)+C$

130. $\frac{1}{2}x+\frac{1}{4}\sin 2x+C$

131. $\frac{1}{4}\operatorname{tg}4x-x+C$

132. $-\frac{2}{3}(\cos x)^{\frac{3}{2}}+C$

133. $-\frac{1}{3\sin^3 x}+C$

134. $\cos x-2\operatorname{arctg}(\cos x)+C$

135. $\frac{1}{3}\sin^3 x-\frac{1}{5}\sin^5 x+C$

136. $\frac{1}{3}\cos^3 x-\cos x+C$

137. $\frac{3}{8}(x-\sin x\cos x-\frac{2}{3}\sin^3 x\cos x)+C;\ \ \sin^2 x=\frac{1-\cos 2x}{2}$

138. $\frac{1}{8}(x-\frac{1}{4}\sin 4x)+C$

139. $\frac{1}{2}\operatorname{tg}^2 x+\ln|\cos x|+C;\ \ \int\operatorname{tg}^3 x dx=\int\operatorname{tg}x\frac{1}{\cos^2 x}dx-\int\operatorname{tg}x dx$

140. $\frac{1}{16}x-\frac{1}{64}\sin 4x+\frac{1}{48}\sin^3 2x+C$

141. $\frac{1}{2}\operatorname{arctg}\left(2\operatorname{tg}\frac{x}{2}\right)+C;\ \ \operatorname{tg}\frac{x}{2}=t$

142. $\frac{1}{4}\ln\left|\frac{2+\operatorname{tg}\frac{x}{2}}{2-\operatorname{tg}\frac{x}{2}}\right|+C$

143. $\frac{2}{\sqrt{5}}\operatorname{arctg}\frac{2+3\operatorname{tg}\frac{x}{2}}{\sqrt{5}}+C$

144. $\operatorname{arctg}\frac{1+5\operatorname{tg}\frac{x}{2}}{2}+C;\ \ \operatorname{tg}\frac{x}{2}=t$

145. $\frac{1}{8}\ln\left|\frac{(1+\sin x)^2}{(3+\sin x)(1-\sin x)}\right|+C$

146. $\frac{1}{5}\ln\left|\frac{\operatorname{tg}\frac{x}{2}-\frac{1}{2}}{\operatorname{tg}\frac{x}{2}+2}\right|+C$

147. $\sqrt{2}\operatorname{arctg}(\sqrt{2}\operatorname{tg}x)-x+C;\ \ \operatorname{tg}x=t$

148. $\frac{1}{2}\operatorname{arctg}(\sin^2 x)+C;\ \ \sin^2 x=t$

149. $-\frac{2}{3}\sqrt{\left(\frac{1+x}{x}\right)^3}+C;\ \ \frac{1+x}{x}=t^2$

150. $\ln|1-\sqrt{1-x^2}|-\ln|x|-\arcsin x+C$

151. $\frac{3}{8}\left(\frac{1+x}{1-x}\right)^{\frac{4}{3}}+C;\ \ \frac{1+x}{1-x}=t^3$

152. $2\sqrt{x}-3\sqrt[3]{x}+6\sqrt[6]{x}-6\ln(1+\sqrt[6]{x})+C;\ \ x=t^6$

153. $6(\sqrt[6]{x}-\operatorname{arctg}\sqrt[6]{x})+C;\ \ x=t^6$

154. $6\sqrt{x}-\frac{4}{5}x\sqrt[4]{x}+C;\ \ x=t^4$

155. $-\frac{3\sqrt[6]{x}}{1+\sqrt[3]{x}} + 3\operatorname{arctg}\sqrt[6]{x} + C;\ \ x = t^6$

156. $6\left(\frac{\sqrt{x}}{3} + \sqrt[6]{x} + \frac{1}{2}\ln\left|\frac{\sqrt[6]{x}-1}{\sqrt[6]{x}+1}\right|\right) + C;\ \ x = t^6$

157. $\frac{6}{5}\sqrt[6]{(x+1)^5} + 2\sqrt{x+1} - 3\sqrt[3]{x+1} - \ln\left[\sqrt[3]{x+1} - \sqrt[6]{x+1} + 1\right]$
$-4\ln(1+\sqrt[6]{x+1}) + 2\sqrt{3}\operatorname{arctg}\frac{2\sqrt[6]{x+1}-1}{\sqrt{3}} + C;\ x+1 = t^6$

158. $-\frac{5}{2}(3-2x^{\frac{3}{5}})^{\frac{1}{2}} + \frac{5}{18}(3-2x^{\frac{3}{5}})^{\frac{3}{2}} + C;\ \ 3-2x^{\frac{3}{5}} = t^2$

159. $\arcsin\frac{x}{\sqrt{7}} + C$

160. $\ln|x+\sqrt{x^2-3}| + C$

161. $\frac{1}{2}\arcsin(2x-1) + C$

162. $\ln|x+1+\sqrt{x^2+2x-1}| + C$

163. $2\sqrt{x^2+9} + 3\ln(x+\sqrt{x^2+9}) + C$

164. $-\sqrt{16-x^2} - 2\arcsin\frac{x}{4} + C$

165. $2\sqrt{x^2+4x+8} + 2\ln(x+2+\sqrt{x^2+4x+8}) + C$

166. $\frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\arcsin x + C$

167. $\frac{1}{2}x\sqrt{9+x^2} - \frac{9}{2}\ln(x+\sqrt{9+x^2}) + C$

168. $\frac{x-3}{2}\sqrt{x^2-6x-7} - 8\ln|x-3+\sqrt{x^2-6x-7}| + C$

2.2. Całki oznaczone

Obliczyć całki oznaczone:

1. $\displaystyle\int_{-3}^{-2}\frac{dx}{x^2+2x+1}$

2. $\displaystyle\int_0^4 x^3\sqrt{x^2+9}\,dx$

3. $\displaystyle\int_{\frac{\pi}{2}}^{\pi}\frac{\cos x}{\sqrt{1+\sin x}}dx$

4. $\displaystyle\int_1^{e^\pi}\sin(\ln x)dx$

Rozwiązania

1. Ponieważ

$$x^2+2x+1 = (x+1)^2,$$

więc

$$\int_{-3}^{-2}\frac{dx}{x^2+2x+1} = \int_{-3}^{-2}\frac{dx}{(x+1)^2} = -\frac{1}{x+1}\Bigg|_{-3}^{-2} = -\left[-1-\frac{1}{-3+1}\right] = \frac{1}{2}.$$

2. Podstawmy

$$\sqrt{x^2+9} = t,$$

a więc

$$x^2+9 = t^2$$

i stąd

$$2xdx = 2tdt.$$

Wprowadzając zmienną t zmieniamy granice całkowania:

$$t_1 = \sqrt{0+9} = 3, \qquad t_2 = \sqrt{16+9} = 5.$$

Stosując teraz twierdzenie o całkowaniu przez podstawienie, mamy

$$\begin{aligned}\int_0^4 x^3\sqrt{x^2+9}dx &= \int_0^4 xx^2\sqrt{x^2+9}dx = \int_3^5 t^2(t^2-9)dt\\ &= \int_3^5(t^4-9t^2)dt = \left(\frac{t^5}{5}-3t^3\right)\Bigg|_3^5\\ &= \frac{1}{5}(5^5-3^5)-3(5^3-3^3) = \frac{1412}{5}.\end{aligned}$$

3. Podstawiamy

$$\sqrt{1+\sin x} = t,$$

a więc

$$1+\sin x = t^2,$$

oraz

$$\cos x dx = 2tdt.$$

Nowe granice całkowania są:

$$t_1 = \sqrt{1+\sin\frac{\pi}{2}} = \sqrt{2}, \qquad t_2 = \sqrt{1+\sin\pi} = 1.$$

Na podstawie twierdzenia o całkowaniu przez podstawienie, otrzymamy

$$\int_{\frac{\pi}{2}}^{\pi}\frac{\cos x}{\sqrt{1+\sin x}}dx = 2\int_{\sqrt{2}}^1 dt = 2t\Big|_{\sqrt{2}}^1 = 2(1-\sqrt{2}).$$