Process of convergence in EU

The Solow growth model

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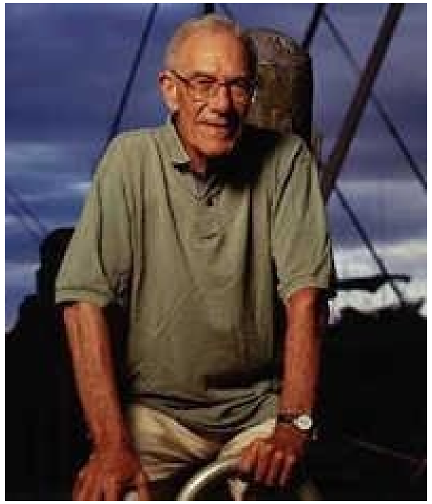
*Slides adapted from teaching materials produced by dr hab. Joanna Wolszczak-Derlacz

Robert Solow

"A Contribution to the Theory of Economic Growth."
Quarterly Journal of Economics 70 (February 1956): 65-94.



Nobel Prize - 1987

Solow neoclassical growth model

- Analysis of (mainly) medium term determinants and effects of economic growth process
- Assumes that **capital to labor ratio** (K/L=k) is the key factor influencing the growth process
- Uses Cobb-Douglas production function as a basis for the analysis

The **Solow Growth Model** is designed to show how growth in the capital stock, growth in the labor force, and advances in technology interact in an economy, and how they affect a nation's total output of goods and services.

Mankiw, 2013

Neoclassical production function (no technology)

$$Y(t) = F\big(K(t), L(t)\big)$$

$Y(t)$ is the total amount of production of the final good at time t

$K(t)$ is the capital stock

$L(t)$ is the labour force

Basic chacacteristics of neoclassical production function

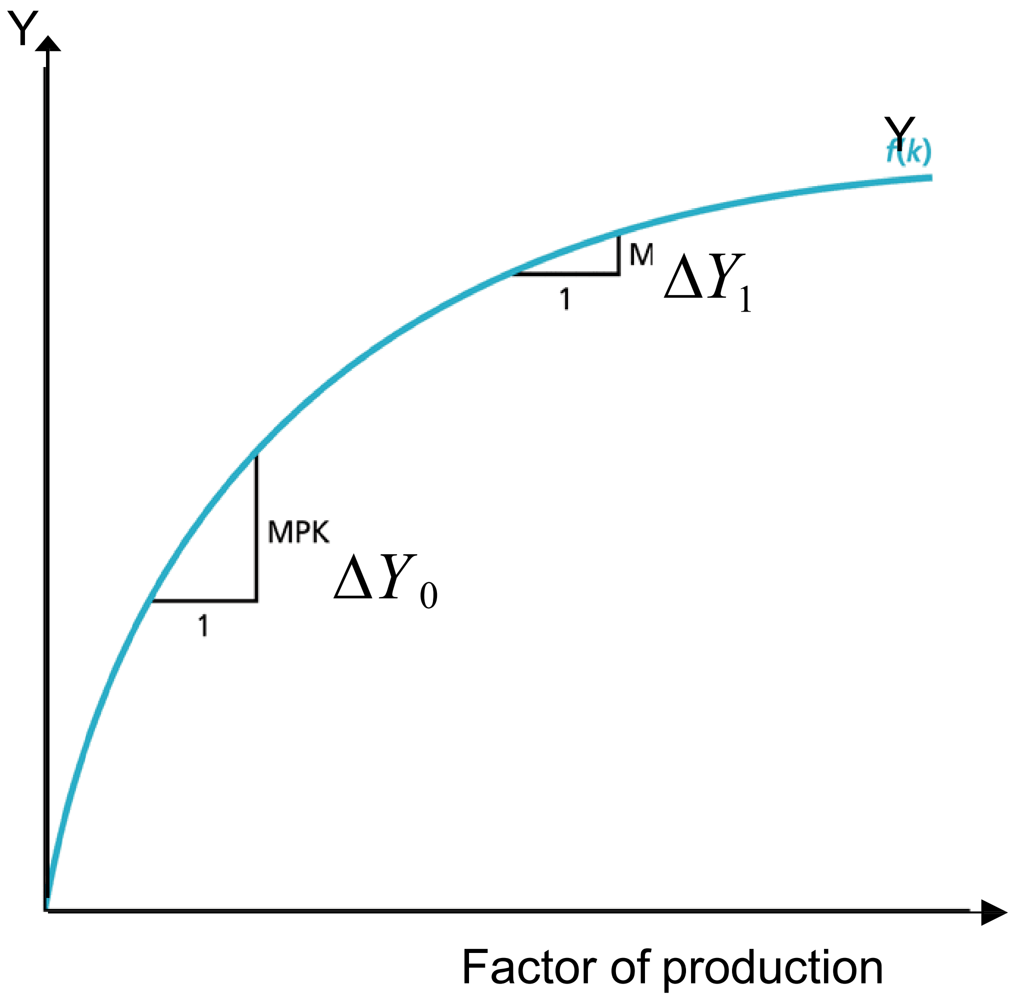
1. Diminishing returnes to labour and capital separately

$$\frac{\partial F}{\partial K} > 0 \qquad \frac{\partial^2 F}{\partial K^2} < 0$$

$$\frac{\partial F}{\partial L} > 0 \qquad \frac{\partial^2 F}{\partial L^2} < 0$$

2. Constant returns to scale

$$F(\alpha K, \alpha L) = \alpha F(K, L)$$



Assumptions

1. Factor of production: labour (L) and capital (K)
2. The economy is closed without government intervantion
3. From the income flow : $I_t = S_t$ and if $S_t = sY_t$ ➡ $I_t = sY_t$
4. Depreciation rate is constant:

$$\delta = \frac{Depreciaion}{Capital} = \frac{D}{K}$$

5. Population growth rate $\frac{\Delta L}{L} = n$

Investment: expenditure on plant and equipment.

Depreciation: wearing out of old capital; causes capital stock to fall e.g. If, say, the typical machine lasts for 5 years, then the depreciation rate is 20 percent

The change in the capital

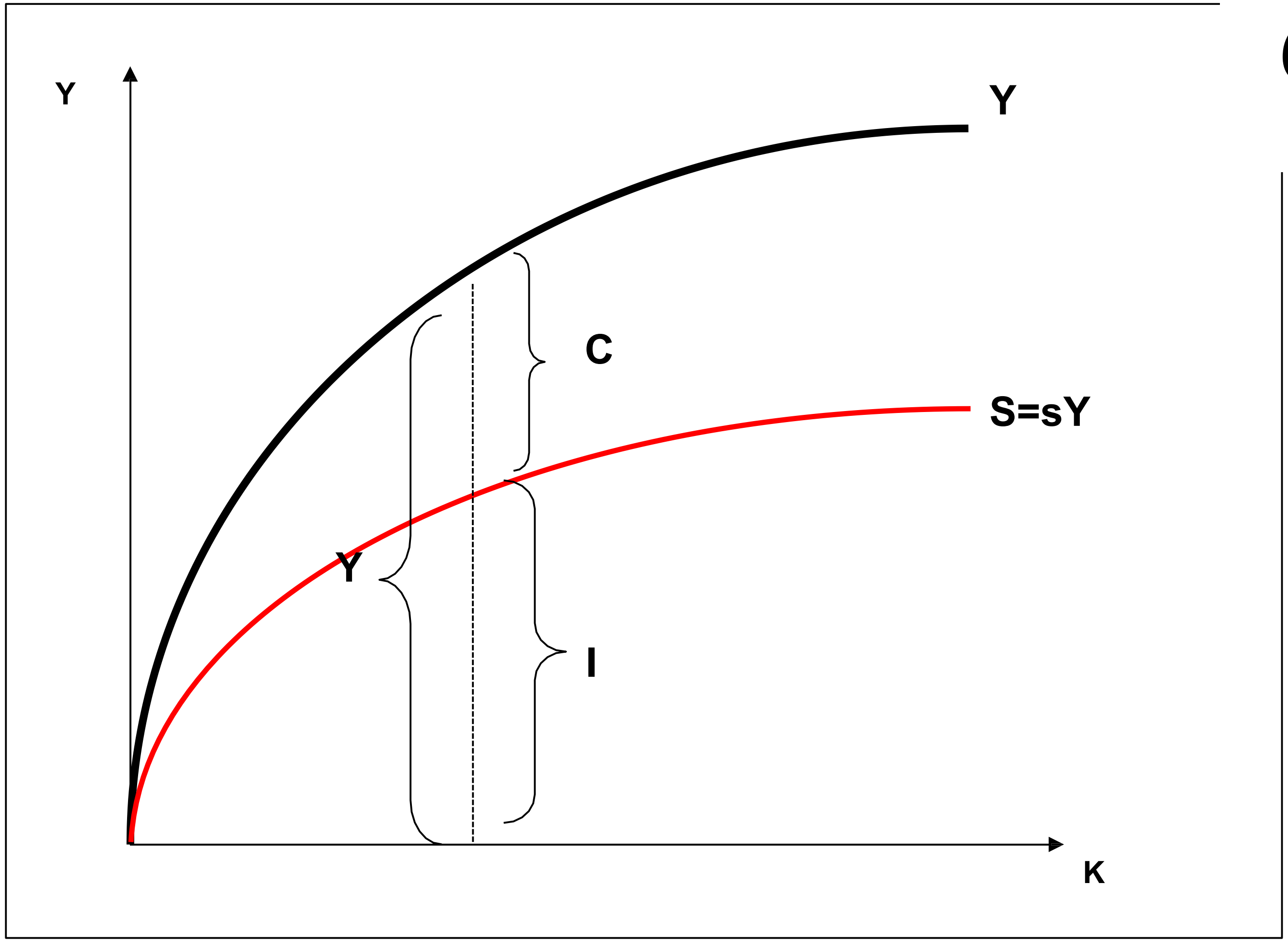
$$\dot{K} = \Delta K = I - \delta K = sY - \delta K \qquad (1)$$

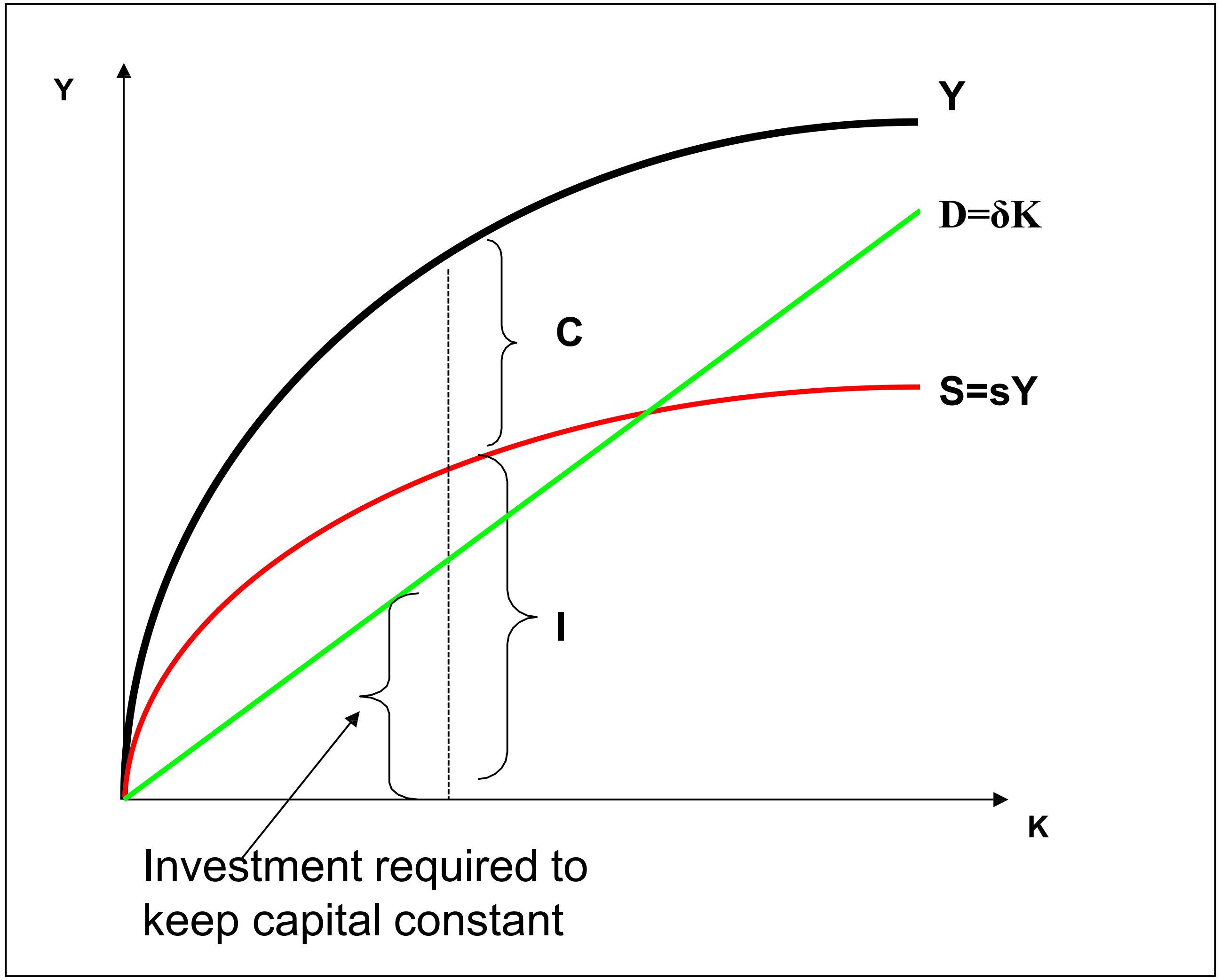
Change in capital stock (↑ ΔK) — Investment (↑ I) — Depreciation (↑ δK)

Neoclassical production function

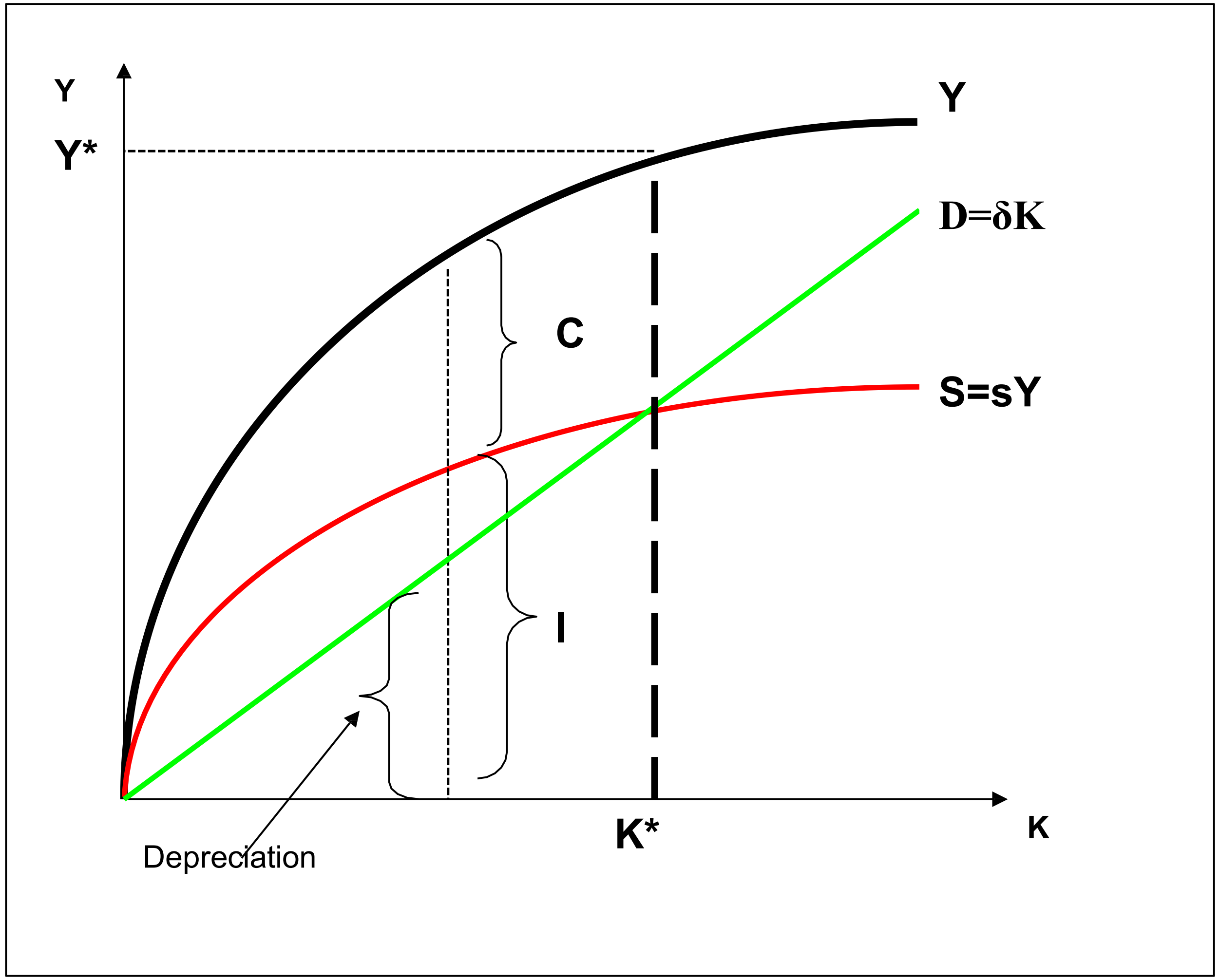
$$Y = C + I$$

$$C = (1-s)Y$$



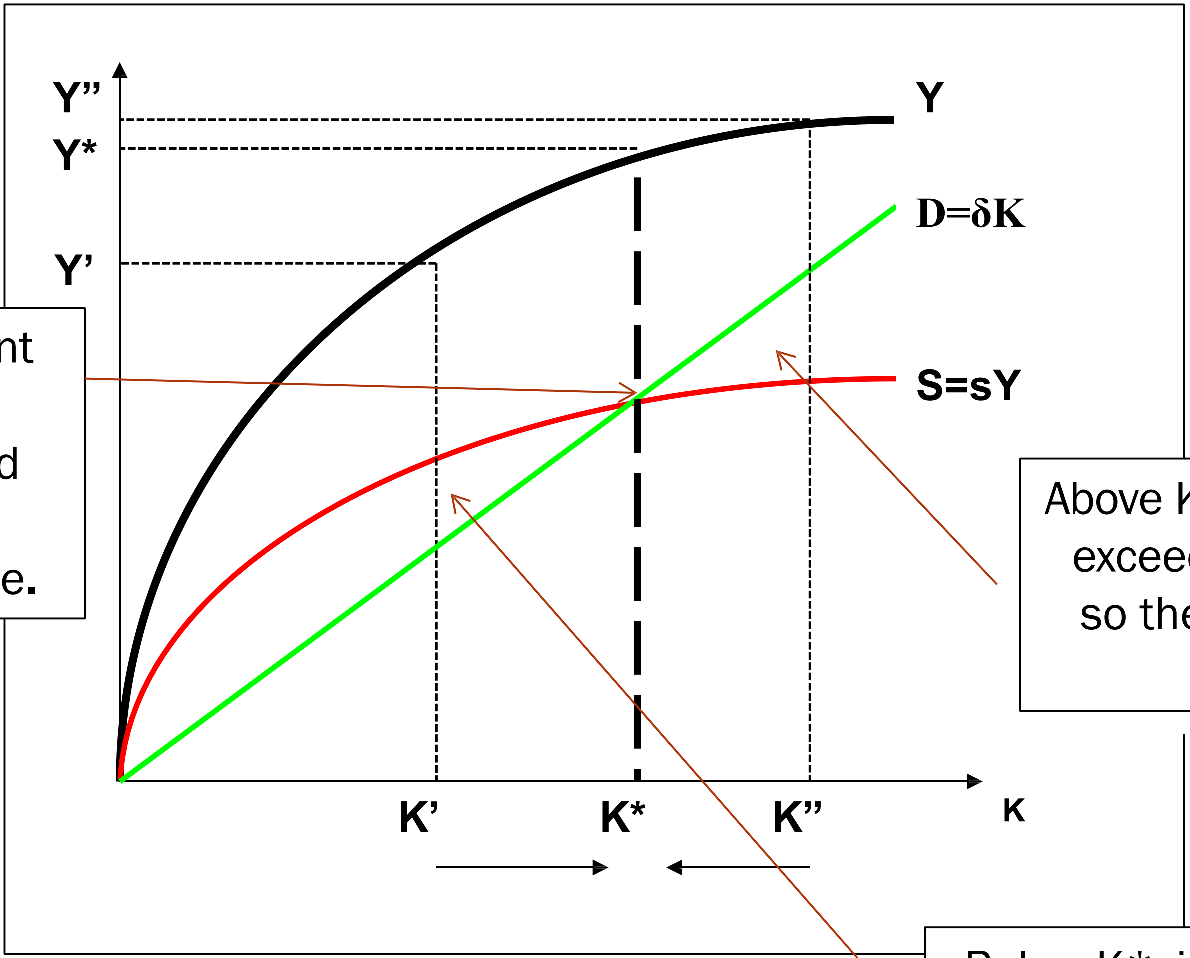

Y
Y
D=δK
C
S=sY
I
K
Investment required to keep capital constant

Steady state



$$\Delta K = 0 \Rightarrow sY = \delta K$$

Steady state



At K^*, investment equals depreciation and capital will not change over time.

Above K*, depreciation exceeds investment, so the capital stock shrinks.

Below K*, investment exceeds depreciation, so the capital stock grows.

$$\Delta K = 0 \Rightarrow sY = \delta K$$

$$sY > \delta K \Rightarrow \Delta K > 0$$ Capital stock grows

$$sY < \delta K \Rightarrow \Delta K < 0$$ Capital stock shrinks

The per worker production funtion:

$$k(t) = \frac{K(t)}{L(t)}$$ Capital per worker (2)

$$y = \frac{Y(t)}{L(t)}$$ Output per worker (3)

Trick – „take logs and then derivatives”

Taking logs and differentiating expression (2) with respect to time, we obtain:

$$\ln k(t) = \ln\left(\frac{K(t)}{L(t)}\right) = \ln K(t) - \ln L(t)$$

$$\frac{d\left(\ln k(t)\right)}{dt} = \frac{d\left(\ln(\frac{K(t)}{L(t)}\right)}{dt} = \frac{d\left(\ln(K(t)) - \ln(L(t))\right)}{dt} \Rightarrow \qquad (4)$$

$$\frac{\dot{k}(t)}{k(t)} = \frac{\dot{K}(t)}{K(t)} - n$$

Substituting for $\dot{K}(t)$ from equation (1) we derive:

$$\frac{\dot{k}(t)}{k(t)} = \frac{sY(t) - \delta K(t)}{K(t)} - n = \frac{sY(t)}{K(t)} - \delta - n =$$

$$= \frac{syL}{kL} - (\delta + n) = \frac{sy}{k} - (\delta + n)$$

$$\boxed{\dot{k} = sy - (\delta + n)k} \qquad (5)$$

,

The steady-state is a condition of the economy in which output per worker (Productivity of labour) and capital per worker (Capital intensity) do not change over time. This is due to the rate of new capital production from invested savings exactly equalling the rate of existing capital depreciation

We define steady state by the condition that $\dot{k}(t) = 0$ then setting equation (5) to zero

$$\boxed{sy = (\delta + n)k}$$

Finding the steady state value of capital per worker and output per worker

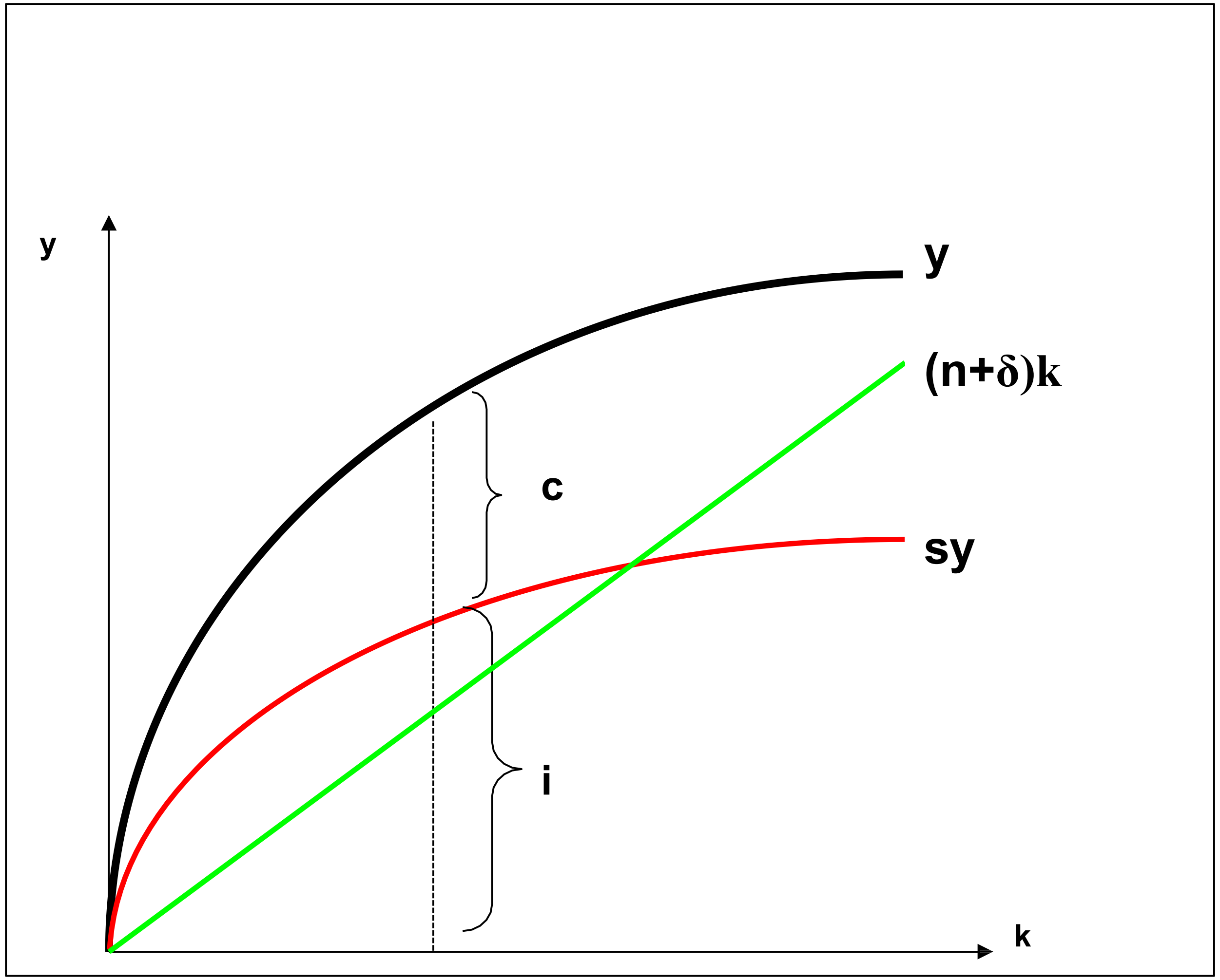
$$\boxed{sy = (\delta + n)k}$$

$$y = \frac{Y(t)}{L(t)} = \frac{F(K,L)}{L} = F(\frac{K}{L,}\frac{L}{L}) = \left(\frac{K}{L}\right)^{\alpha}\left(\frac{L}{L}\right)^{1-\alpha} = k^{\alpha}$$

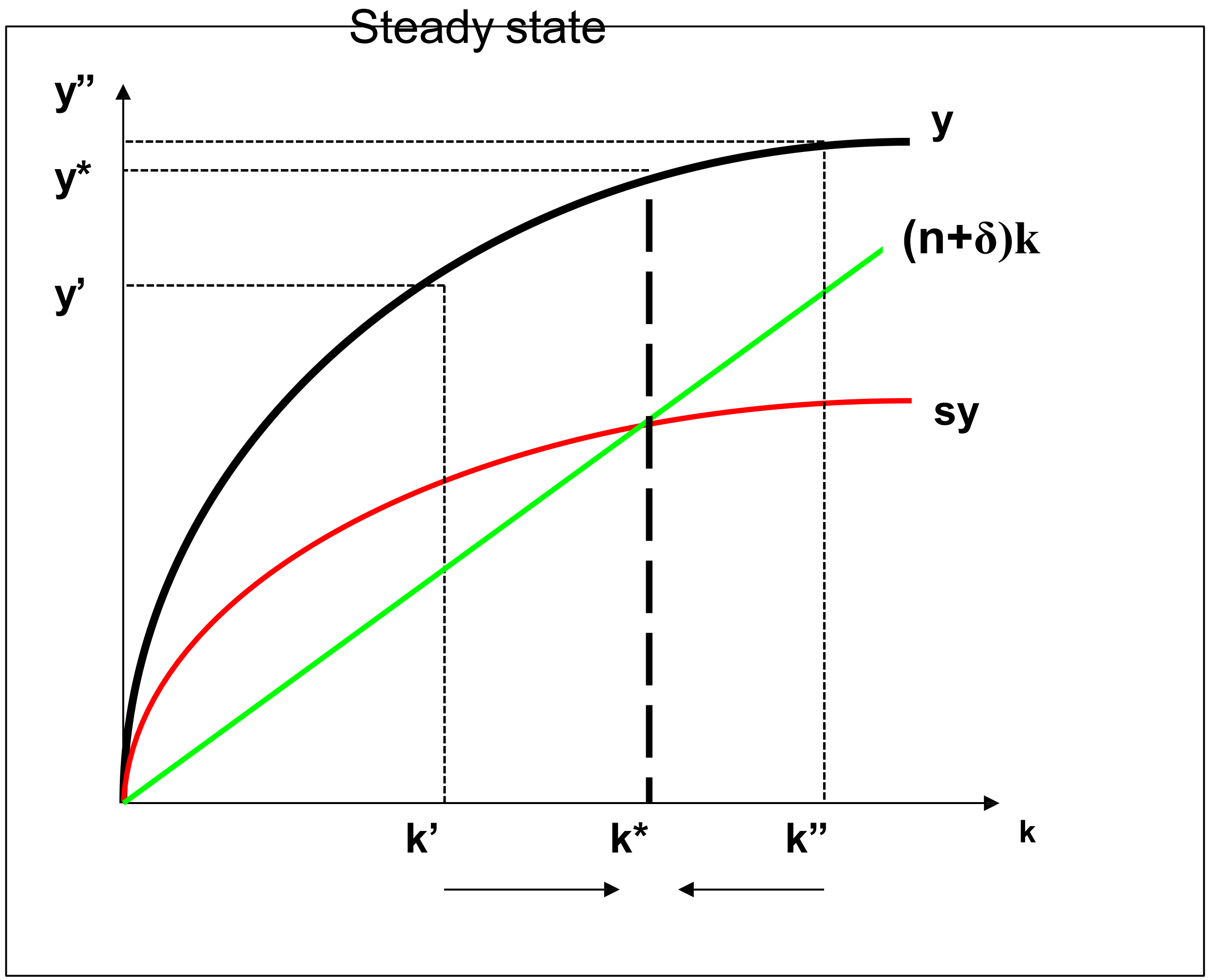
$$k^{ss} = \left(\frac{s}{\delta + n}\right)^{1/1-\alpha}$$

$$y^{ss} = \left(\frac{s}{\delta + n}\right)^{\alpha/1-\alpha}$$

Per worker production function



Like depreciation, population growth is one reason why the capital stock per worker shrinks. If n is the rate of population growth and δ is the rate of depreciation, then $(n+\delta)k$ is **break-even investment,** which is the amount necessary to keep constant the capital stock per worker *k*.

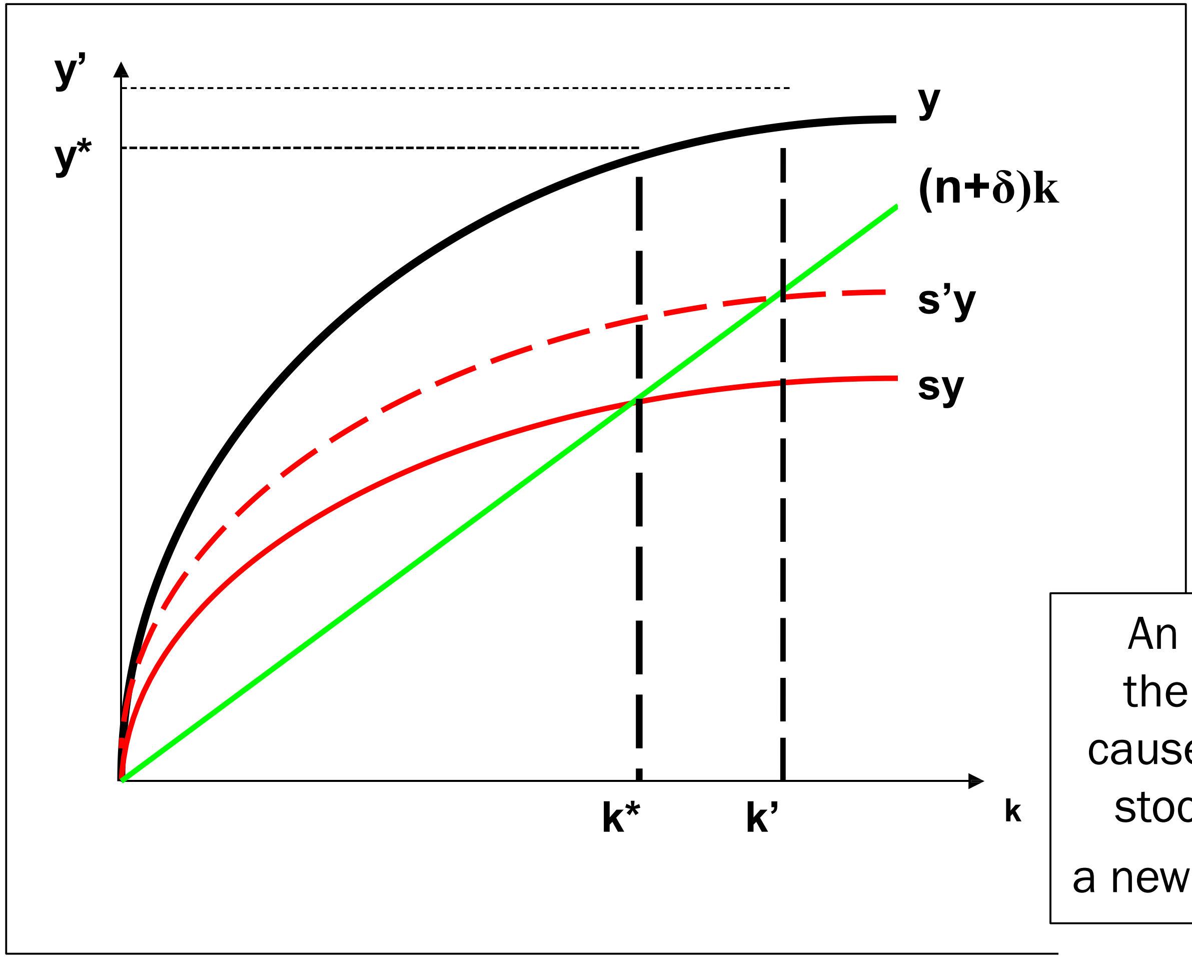


In the steady state, the positive effect of investment on the capital per worker just balances the negative effects of depreciation and population growth. Once the economy is in the steady state, investment has two purposes:

1) Some of it, (δk^*), replaces the depreciated capital,

2) The rest, (nk^*), provides new workers with the steady state amount of capital.

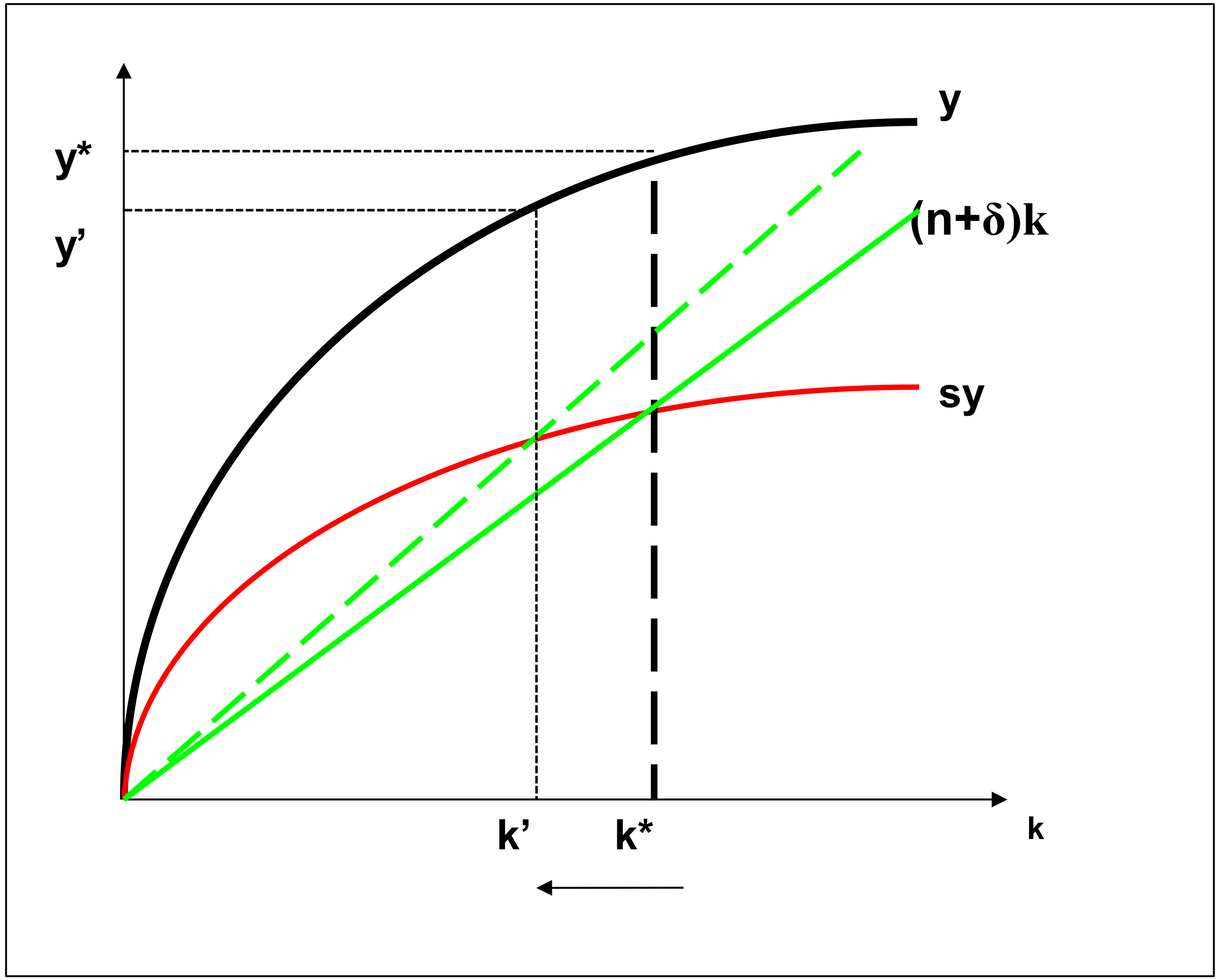
An increase in savings



An increase in the saving rate causes the capital stock to grow to a new steady state.

Increase in *s* rises *k** and *y**

Population growth change



Increase in population growth n lowers *k** and *y**

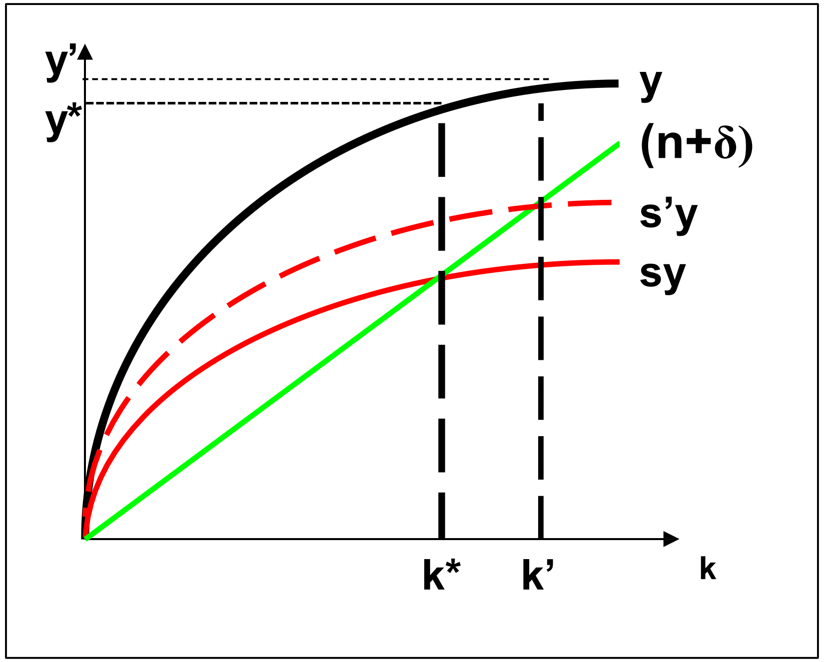
Change in the rate of population growth

- An increase in the rate of population growth shifts the line representing population growth and depreciation upward.

- The new steady state has a lower level of capital per worker than the initial steady state.

- Thus, the Solow model predicts that economies with higher rates of population growth will have lower levels of capital per worker and therefore lower incomes.

Savings in Solow model



Change in the savings rate

- In the long run, an economy's saving determines the size of *k* and thus *y.*
- The higher the rate of saving, the higher the **stock** of capital and the higher the level of *y.*
- An increase in the rate of saving causes a period of rapid growth, but eventually that growth slows as the new steady state is reached.
- **Conclusion**: although a high saving rate yields a high steady-state level of output, **saving by itself cannot generate persistent economic growth.**

What should be the saving rate?

- The effect of increse in *s* is the rise of k and y at the steady state
- The effect of increse in *s* is that lower part of income is consumed (Y=S+C)
- If we save all income → consumption =0,
- If we consume all income → saving=0

What is the savings rate *s** that maximizes the steady state level of aggregate consumption *C* * per unit of effective labour?

The answer is the so called "golden rule"

Golden saving rule

Condition for steady state

$$\Delta k = 0 \quad (1)$$

$$\Delta k = sy - \delta k \quad (2)$$

$$0 = sy - \delta k \quad (3)$$

$$sy = \delta k \quad (4)$$

$$C = y - sy = k^{\alpha} - \delta k \quad (5)$$

$$\frac{dC}{dk} = \alpha k^{\alpha-1} - \delta \quad (6)$$

$$\frac{dC}{dk} = 0 \quad (7)$$

$$\alpha k^{\alpha-1} - \delta = 0 \quad (8)$$

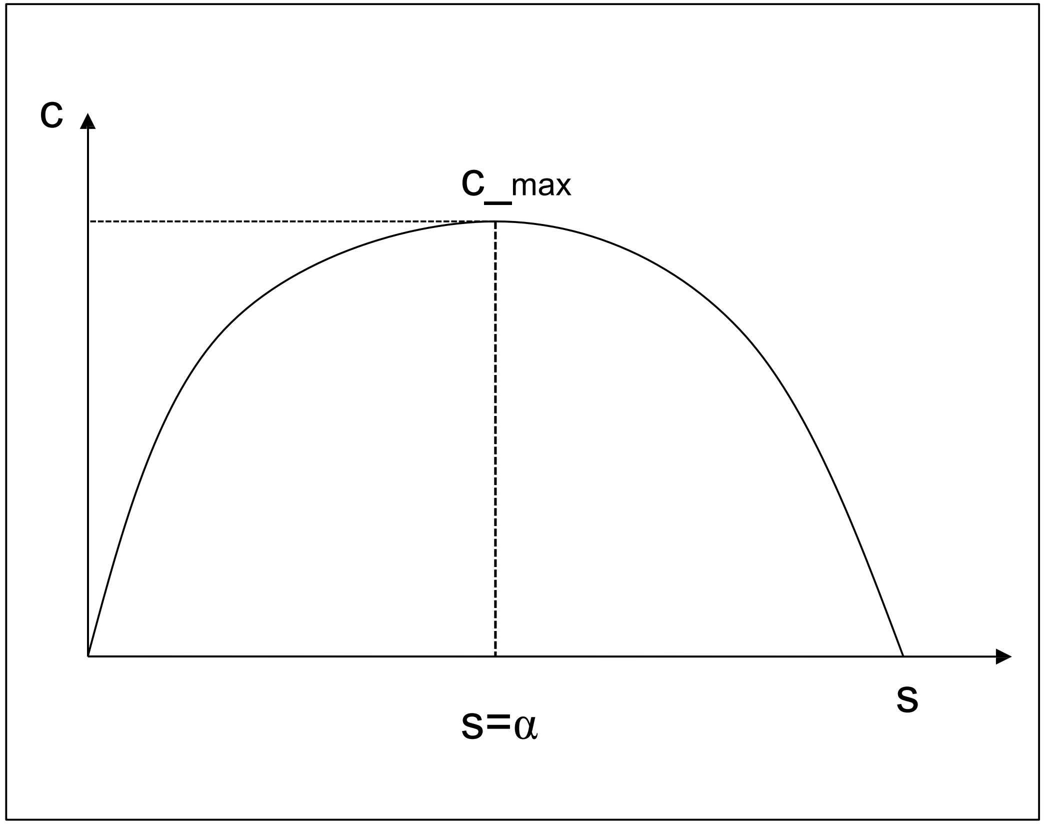
$$\alpha k^{\alpha-1} = \delta \quad (9)$$

Coming back to the steady state condition (4) and putting (9)

$$sy = \delta k = \alpha k^{\alpha-1} \times k = \alpha k^{\alpha} = \alpha y$$

divide by y → $s = \alpha$

Golden saving rule



Appendix – some Mathematics

$$\Delta X = X_{t+1} - X_t$$

$$\frac{\Delta X}{X} = \frac{X_{t+1} - X_t}{X_t} = \frac{\dot{X}}{X}$$

„percentage change"

$$\frac{dX}{dt} = \lim_{\Delta t \to 0} \frac{X_{t+\Delta t} - X_t}{\Delta t} = \dot{X}$$

$$\dot{L}/L = \frac{dL(t)/dt}{L}$$

e.g. =0.02 it means that the labor force is growing at 2 percent per year

Natural logs

$$z = xy \Rightarrow \ln z = \ln x + \ln y$$

$$z = \frac{x}{y} \Rightarrow \ln z = \ln x - \ln y$$

$$z = x^{\beta} \Rightarrow \ln z = \beta \ln x$$

$$y = f(x) = \ln x, \Rightarrow \frac{dy}{dx} = \frac{1}{x}$$

$$y(t) = \ln x(t), \Rightarrow \frac{dy}{dx} = \frac{dy}{dx}\frac{dx}{dt} = \frac{1}{x}\dot{x} = \frac{\dot{x}}{x}$$

Derivative with respect to time of the log of some variable is the growth rate of that variable

$$d\ln(X)/dt = (1/X)dX/dt = \frac{\dot{X}}{X}$$

"Take logs and derivatives"

$$Y = K^{\alpha} L^{1-\alpha}$$

$$\ln Y = \ln K^{\alpha} + \ln L^{1-\alpha}$$

$$\ln Y = \alpha \ln K + (1-\alpha) \ln L$$

$$\frac{d \ln Y}{dt} = \alpha \frac{d \ln K}{dt} + (1-\alpha) \frac{d \ln L}{dt}$$

$$\frac{\dot{Y}}{Y} = \alpha \frac{\dot{K}}{K} + (1-\alpha) \frac{\dot{L}}{L}$$

Sources:

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- Acemoglu D. Growth Theory Since Solow and the Poverty of Nations.
- World Bank, April 26, 2006http://econ-www.mit.edu/files/970
- Mankiew N.G. (2012), Macroeconomics, 8th Edition, Worth Publishers