Healing, Super Healing, and Other Issues in Continuum Damage Mechanics

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Abstract

In this chapter, after a quick introduction on the literature of healing and super healing concept, the damage/healing mechanics principles are investigated. The concept of super healing of materials is then introduced into the framework of continuum damage mechanics (CDM). Super-healed material can be seen as a strengthened material by further healing when the whole damage is recovered by healing of a damaged material. Therefore, in this chapter the process of healing beyond what is necessary for damage recovery is called super healing. Super material is the final objective of the super healing process when the material

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achieves more stiffness at the end of super healing process. Then, by introducing the anisotropic super healing concept, these concepts are generalized in tensorial form to be used in anisotropic damage and healing of materials. Consequently, three fundamental issues in CDM are discussed. Nature of the damage process is investigated by dissecting the expression of the effective stress into an infinite geometric series. Several stages of damage are introduced which are termed primary, secondary, tertiary, etc., using this expression. New definition of the damage variable is then introduced for small damage cases. The new concept of undamageable materials that maintain a zero value of the damage variable throughout the deformation process is introduced. Finally and in the last section of the chapter, the forming of a singularity which leads to initiation of the process of fracture is shown in a continuous region within the framework of CDM. The internal damage processes leading to a singularity are illustrated mathematically. This section potentially provides a crucial link between the damage and fracture mechanics.

Introduction

Self-healing process in the material from damaged state has paid attention increasingly in the damage mechanics literature. Undergoing research shows two different self-healing mechanisms: one is an active or autonomous system as a coupled system in which damaging triggers self-healing (Pang and Bond 2005; Toohey et al. 2007; White et al. 2001); another is passive system as a decoupled system in which healing occurs after damage identification using external detection (John and Li 2010; Li and John 2008; Li and Muthyala 2008; Li and Nettles 2010; Li and Uppu 2010; Liu and Chen 2007; Nji and Li 2010a, b; Varley and van der Zwaag 2008; Zako and Takano 1999). Some significant new results on damage and healing of materials are presented by Pavan et al. (2010), Yuan and Ju (2012), and Zaïri et al. (2011). This process is observed experimentally even in nanoscale in wide range of materials (George and Warren 2002; Nemat-Nasser 1979, 1983; Voyiadjis and Park 1996; Wang and Sekerka 1996). Healing in constitutive models is used in two different approaches: one is used to characterize the healing process usually by a phenomenological approach (Miao et al. 1995), and the other is simple model (Adam 1999; Simpson et al. 2000). Thermodynamic-based damage and healing models are also introduced recently and are available in the literature (Barbero et al. 2005; Miao et al. 1995). However, constitutive modeling of self-healing material is still in progress since experimental aspect of healing has significant difficulties. During the past decades, progress has been made in the damage mechanics of various materials including elastoplastic models (Chaboche 1991; Ginzburg 1955; Kattan and Voyiadjis 1993; Lee et al. 1985; Naderi et al. 2012; Voyiadjis 1988; Voyiadjis and Kattan 1990, 1992; Voyiadjis et al. 2012); elasto-viscoplastic models (Chaboche 1997; Lemaitre and Chaboche 1990); continuum damage models (Kachanov 1958; Voyiadjis and Kattan 2009); materials surface degradation models including rolling, sliding contact fatigue, fretting fatigue, and adhesive wear (Loginova et al. 2001; Singer-Loginova and Singer 2008; Wheeler et al. 1993); and coupled elastoplastic damage models (Chow and Jie 2009; Lemaitre 1985; Voyiadjis et al. 2009). The damage variable in scalar or second-order tensor forms shows the average material degradation (loss of stiffness). This variable lumps all kinds of defects such as micro-cracks, voids, and micro-cavities at the microscale level (Lubarda and Krajcinovic 1993; Voyiadjis and Kattan 2009). It is shown that in the case of isotropic damage, two independent damage variables are necessary to predict damage level (Cauvin and Testa 1999; van der Waals 1979). It has been argued that sufficient accuracy can be obtained to find certain parameters of damaged materials under the assumption of isotropic damage (Lemaitre 1984).

The representative volume element (RVE) is widely used in continuum damage mechanics in which the discontinuities (micro-voids, micro-cracks, etc.) are not considered explicitly in the RVE. The discontinuous and discrete elements of damage effects are lumped together through the use of a macroscopic internal variable. Phenomenological approach is adopted usually, and the consistent formulation is derived using sound mechanical and thermodynamic principles. Thermodynamically consistent framework is achieved using the concept of macroscopic internal variables which is used to lump the effect of all defects (Ginzburg and Landau 1965; Hansen and Schreyer 1994; Landau and Ter Haar 1965; Miao et al. 1995; Murakami 1983; Voyiadjis and Park 1997; Voyiadjis and Kattan 2006, 2012b; Voyiadjis and Park 1995; Voyiadjis et al. 2009). The concept of effective stress for uniaxial tension was first introduced by Kachanov (1958) and Rabotnov (1968). It has been shown that the isotropic damage assumption is sufficiently enough to predict the load carrying capacity and the number of cycles or the time to local failure in structural components (Kattan and Voyiadjis 2001; Voyiadjis and Kattan 2005, 2006). However, anisotropic damage propagation has been observed experimentally (Lee et al. 1985; Sidoroff 1981) even in an initially isotropic solid. The damage variable is considered in scalar form in the case of isotropic damage mechanics, and the evolution equations can be handled easily (Voyiadjis and Kattan 2009). The concept of an undamageable material was proposed recently by Voyiadjis and Kattan (2012b, 2013c, d). This kind of material is considered as a hypothetical material that cannot be damaged during the loading process. Furthermore, decomposition of the damage tensor into two damage components, one due to cracks and one due to voids, is developed by Kattan and Voyiadjis (2001). Finally, a conceptual framework for general damage processes operating in series and in parallel is introduced by Voyiadjis and Kattan (2012a).

In this chapter, after a quick introduction on the literature of healing and super healing concept, the damage/healing mechanics principles are investigated. The concept of super healing of materials is then introduced into the framework of continuum damage mechanics. Super-healed material can be seen as a strengthened material by further healing when the whole damage is recovered by healing of a damaged material. Therefore, in this chapter the process of healing beyond what is necessary for damage recovery is called super healing. Super material is the final objective of the super healing process when the material achieves more stiffness at the end of super healing process. Consequently, by introducing the anisotropic super healing concept, these concepts are generalized in tensorial form to be used in anisotropic damage and healing of materials. Mechanics of damage/healing and super healing are illustrated using an example for the case of plane stress. Consequently, the characteristics of the super material are outlined in the following section. Furthermore, three fundamental issues in continuum damage mechanics are discussed, and fracture mechanics and damage mechanics are linked through the internal damage process leading to singularity in continuous regions.

Review of Damage/Healing Mechanics

In this section, the damage/healing mechanics principles are reviewed considering a fictitious undamaged material configuration as shown in Fig. 1. Any nonzero component of the effective Cauchy stress tensor $\overline{\sigma}_{ij}$ such as the tension component $\overline{\sigma}$ can be obtained by the following relation (Kachanov 1958; Rabotnov 1963; Sidoroff 1981; Voyiadjis and Kattan 2006, 2009):

$$\overline{\sigma} = \frac{\sigma}{1 - \varphi} \tag{1}$$

σ is the corresponding component of the Cauchy stress tensor and φ is the isotropic damage variable. The damage variable φ changes between 0 and 1. It is worth to mention that the value $\varphi = 0$ shows the undamaged state while complete failure (fracture) happens when the value of φ tends to 1.

Defining an intermediate configuration as partially healed material which is indicated in Fig. 2 and considering this configuration as a combination of damage and healing between undamaged and damaged state as shown in Fig. 3, the effective stress $\overline{\sigma}$ is written as (Chow and Wang 1987; Park and Voyiadjis 1998; Voyiadjis and Park 1997; Voyiadjis et al. 2012)

$$\overline{\sigma} = \frac{\sigma}{1 - \varphi(1 - h)} \tag{2}$$

where h is the healing variable. The healing variable h is also changes 0 and 1. It is worth to mention that the value $h = 0$ shows the absence of healing; therefore, Eq. 1 can be obtained from Eq. 2 by substituting $h = 0$ in this case. On the other hand, the value $h = 1$ describes complete healing, i.e., recovery of all the damage. In this case, material goes back to the beginning of the loading (undamaged state), and the actual stress and the effective stress are equal in Eq. 2. Comparison between Eqs. 1 and 2 confirms the replacement of the damage variable φ in Eq. 1 by the variable φ $(1 - h)$ in Eq. 2 in self-healing materials. Both damage and healing effects are combined through this new variable in one single parameter

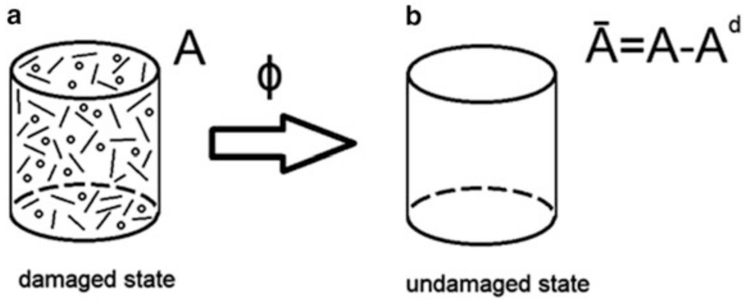

Fig. 1 Damage configurations in terms of cross-sectional area reduction (After Voyiadjis and Kattan 2013a)

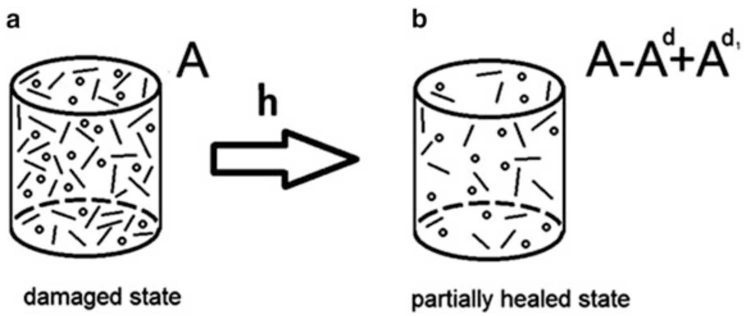

Fig. 2 Healing configurations in terms of cross-sectional area reduction (After Voyiadjis and Kattan 2013a)

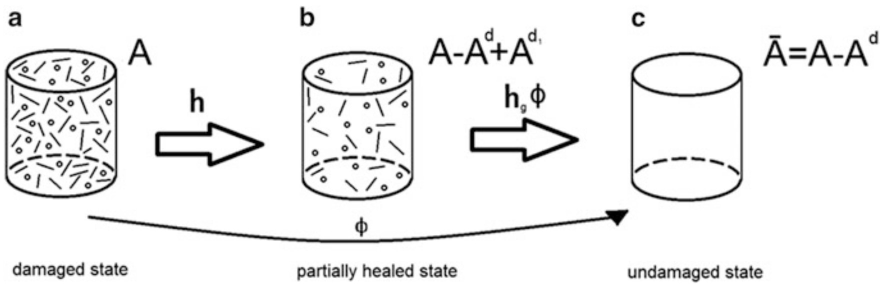

Fig. 3 Damage and healing configurations in terms of cross-sectional area reduction (After Voyiadjis and Kattan 2013a)

which is called here the combined damage/healing variable (Chow and Wang 1987) termed this variable as the effective damage variable. However, Eq. 2 is not written explicitly by Park and Voyiadjis (1998) and Voyiadjis and Park (1997), and it is not recognized as the combined damage/healing variable as a single parameter.

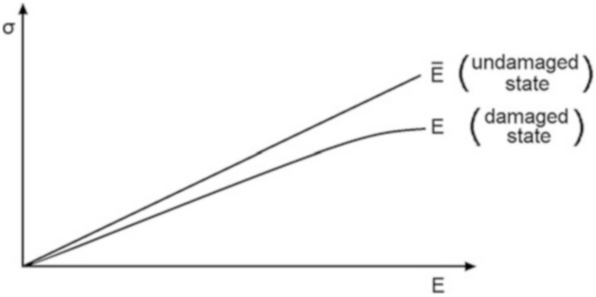

Fig. 4 Damage states in terms of elastic stiffness degradation (After Voyiadjis and Kattan 2013a)

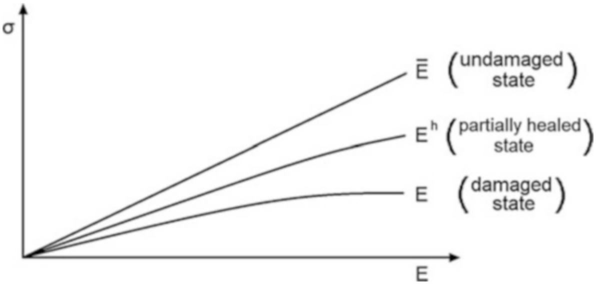

Fig. 5 Damage and healing states in terms of elastic stiffness degradation (After Voyiadjis and Kattan 2013a)

Based on Eq. 2, the complete failure occurs when the value of the new variable (combined damage/healing parameter $\varphi\ (1 - h)$) tends to 1. Two different situations can occur in the absence of damage when the new variable is zero: first one is when $\varphi = 0$ (undamaged virgin material), and the other one is when $h = 1$ (completely healed material). Schematic stress–strain curves are shown in Figs. 4 and 5 for damaged and damaged/healed materials in elastic region, respectively. It is worth to mention that reformulation of the damage and healing principles (Eqs. 1 and 2) in terms of the elastic stiffness as shown in Figs. 4 and 5 is possible, but this is beyond the scope of this chapter.

Experimental test results on bituminous materials are shown in Figs. 6 and 7. These experiments show the capabilities of self-healing materials and the relationship between the healing time and the healing percentage which were conducted by Murray et al. (1995). In their work, Qiu et al. elucidated several characteristics of bituminous materials as related to healing and damage.

It should be noted that the healing parameter h corresponds exactly to the healing percentage which are shown in Figs. 6 and 7. The experimental results fully conform to the theoretical damage/healing mechanics framework which is summarized in this section. Upon further examination of Fig. 7, it is finally clear that the healing effect is limited in the first 10 h, but healing improves significantly during the 100th and 1,000th hours. Certain biological materials like bones can be considered as other applications on healing of materials.

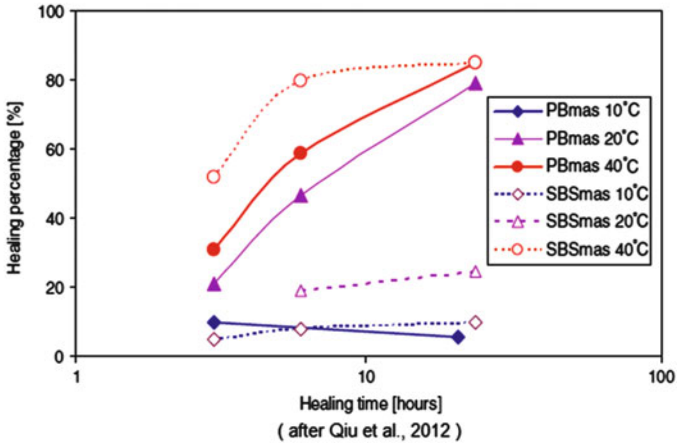

Fig. 6 A practical example of healing time versus healing percentage (After Voyiadjis and Kattan 2013a)

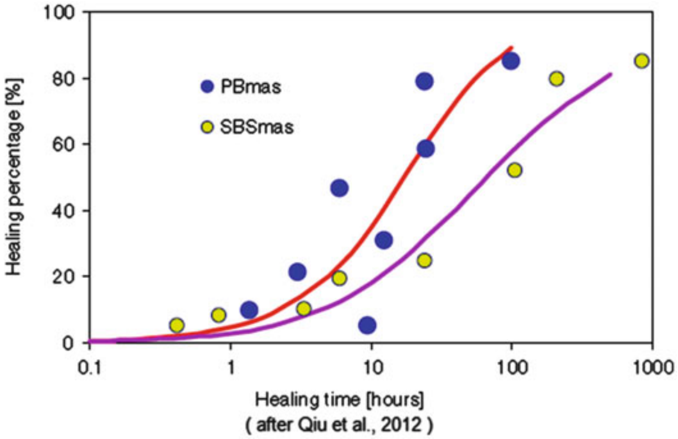

Fig. 7 The relation of healing time to healing percentage in a practical example (After Voyiadjis and Kattan 2013a)

Introduction to Super Healing

In classical damage/healing mechanics, the healing parameter h changes between 0 and 1 in which $h = 0$ implies zero healing and $h = 1$ implies complete healing. Assume that after complete damage recovery, i.e., when $h = 1$, the healing process continues beyond $h = 1$. This will enable us to use large values for h, i.e., h is allowed to take values like 2, 3, 4, …... $n + 1$. This special case is called super healing in which some form of strengthening or enhancing the properties of the material occurs instead of further healing since the material is now undamaged (was completely healed when healing variable reaches to 1 $h = 1$). The values of n are limited to integer values only in the following derivation, and it is not allowed to take real values at this stage. The details for this issue are outlined in section "Characteristics of the Super Material." Therefore, in the super healing phase, the values of the healing parameter h start at 1 and increase to take the values 2, 3, 4, ... n + 1. It is obvious that by high increasing of the value of the healing parameter h, a strengthened material will be obtained. Hypothetical material which is called here the super material is postulated as the value of h approaches infinity. Full characterization of the super material is not the intention of this chapter, but only the way to its realization is pointed out. Some of the characteristics of the super material are postulated in section "Characteristics of the Super Material." It is hoped that future technology will be able to manufacture strong and self-healing materials that come as close as possible to the proposed theoretical super material. Establishing governing equations for the proposed super healing materials is the main aim of this chapter. In this work, what interests us is the mechanics of the process of super healing and not the final super material which remains theoretical and hypothetical at this time.

Equation 2 changes to the following form by assuming the value of the healing parameter h is increased through super healing to n + 1.

$$\overline{\sigma} = \frac{\sigma}{1 + n\varphi} \tag{3}$$

Equation 3 can be considered as the main expression of the process of super healing. Based on Eq. 3, it can be seen that as the value n tends to infinity, the value of the effective stress tends to zero. Thus, the first characteristic of the super material is concluded as its effective stress is zero, and it does not depend on the value of either the damage parameter φ or the healing parameter h. This process is called super healing of order n based on Eq. 3. Super-healed material can be obtained by continuing super healing at different stages as the value of h increases from 1 to 2 to 3 and so on. The super-healed material of order n will be finally obtained when the process of super healing of order n takes its due course.

In damage mechanics, the effective stress becomes infinity which indicates complete rupture of the material when $\varphi = 1$ (see Eq. 1). However, the effective stress does not explode but takes a finite value in super-healed materials when $\varphi = 1$ (see Eq. 3). This means that rupture does not occur in super-healed materials even when the value of the damage parameter approaches to infinity. This can be seen as the significance of the super healing process. Based on Eq. 2, it can be seen clearly that the effective stress explodes when the value of the combined damage/healing parameter φ $(1 - h)$ tends to one. This critical case cannot occur in super-healed materials since by setting $\varphi\,(1 - h) = 1$ the relation for $h = \frac{\varphi - 1}{\varphi}$ is obtained and based on this relation by assuming the value of the damage parameter $\varphi = 1$, then the obtained value for h is $h = 0$ which is not possible. The value of the healing parameter will be negative if the value of the damage parameter is less than 1 which is not possible again. Another possibility is when both the damage and healing parameters take values larger than 1; this case cannot occur since the damage parameter is bounded to be less or equal to 1. From the beginning of this chapter up to this section, a certain mechanism for the super healing process is proposed. By admitting higher values for h larger than 1 up to $n + 1$, the process of super healing is achieved. One single healing mechanism is employed in the first approach and another approach which employs multiple healing mechanisms operating in parallel (i.e., at the same time) for the process of super healing to occur. The equivalency of the two approaches is shown in the following part. Multiple healing parameters h_1, h_2, h_3, ..., h_n characterizing multiple healing mechanisms that are operating at the same time are utilized instead of one single healing parameter h. In this case, the value of each healing parameter is limited to 1 only since there is no need to go beyond 1 in this approach for super healing. Therefore, Eq. 2 can be rewritten in the following form:

$$\overline{\sigma} = \frac{\sigma}{1 - \varphi(1 - h_1 - h_2 - ..... - h_n)} \tag{4}$$

In order to derive Eq. 4, one follows the same procedure used in deriving Eq. 2. For more details, check the references by Chow and Wang (1987), Park and Voyiadjis (1998), and Voyiadjis and Park (1997).

The proposed process of super healing can be achieved by utilizing Eq. 4 characterizing multiple healing parameters (n parameters) operating at the same time and by using the value 1 for each separate healing parameter. Furthermore, the elusive super material is considered again if the number of healing parameters approaches infinity (with each one limited to 1). Equivalency of the two approaches to super healing of Eqs. 2 and 4 is shown in as follows. Resultant healing parameter h is defined as the sum of the multiple healing parameters h_1, h_2, h_3,......., h_n as follows:

$$h = h_1 + h_2 + h_3 + \dots\dots\dots + h_n \tag{5}$$

Therefore, the process of super healing can be obtained in two different ways: first by using one single large healing parameter h with values 1, 2, 3, ... n and second by using multiple small healing parameters h_1, h_2, h_3, ..., h_n operating together but with limit value of one for each parameter. It can be seen that by substituting the value of 1 for each single parameter in Eq. 4, governing equation of super healing (Eq. 3) can be obtained, and thus both approaches are the same. Finally, the following question needs to be asked: What happens by allowing each multiple healing parameter h_1, h_2, h_3, ..., h_n to have a value larger than 1? This will open the way for more healing and super healing. For instance, allowing each healing parameter to take the value of 2, then Eq. 3 reads

$$\overline{\sigma} = \frac{\sigma}{1 + 2n\varphi} \tag{6}$$

This is called Level 2 Super Healing characterized by Eq. 6. In general, the following expression can be obtained which characterizes the process of super healing in the case where each healing parameter is allowed to take the value of n

$$\overline{\sigma} = \frac{\sigma}{1 + n^2\varphi} \tag{7}$$

This is called Level n Super Healing characterized by Eq. 7.

Anisotropic Damage/Healing Mechanics

The theory of damage/healing mechanics of section "Review of Damage/Healing Mechanics" is generalized to anisotropic damage/healing in this section (Chow and Wang 1987; Park and Voyiadjis 1998; Voyiadjis and Park 1997). For this purpose, tensors are used instead of scalars. In the following, capital letters are used to denote fourth-rank tensors, and it is assumed that tensors are represented by matrices. Let M denote the fourth-rank damage effect tensor of continuum damage mechanics. The exact relationship between the fourth-rank damage effect tensor M and the scalar damage variable φ is extensively investigated in the literature (Sidoroff 1981; Voyiadjis and Kattan 2006, 2009).

Let H denote a fourth-rank healing tensor corresponding to the healing parameter h. For the exact relationship between H and h, see Park and Voyiadjis (1998) and Voyiadjis and Park (1997). In the case of anisotropic healing and damage, Eq. 2 can be generalized as follows (Park and Voyiadjis 1998; Voyiadjis and Park 1997):

$$\overline{\sigma}_{ij} = \left[M^{-1}_{ijkl} + \left(I_{ijmn} - M^{-1}_{ijmn} \right) : H^{-1}_{mnkl} \right]^{-1} \sigma_{kl} \tag{8}$$

where I_{ijmn} is the fourth-rank identity tensor. In Eq. 8, σ_{kl} and $\overline{\sigma}_{ij}$ are second-rank stress tensors and are represented by vectors in section "Damage/Healing and Super Healing in Plane Stress." Based on Eq. 8, it is shown that the combined damage/healing parameter $\varphi\,(1 - h)$ is generalized to $(I_{ijmn} - M_{ijmn}{}^{-1})\,(I_{mnkl} - H_{mnkl}{}^{-1})$. The latter expression for combined damage and healing can be derived directly from Eq. 8 considering the facts that the main component of M corresponds to $\frac{1}{1-\varphi}$ and the main component of H corresponds to $\frac{1}{h}$. The fourth-rank healing tensor H clearly satisfies some mathematical properties such as the components of this tensor are either positive or zero, and both the trace and norm of the tensor are positive. However, the tensor H may not necessarily be positive definite. These properties are illustrated with the example of plane stress which is given in section "Damage/Healing and Super Healing in Plane Stress."

Anisotropic Super Healing

The process of super healing for anisotropic damage and healing mechanism is shown in this section. Following the outlined reasoning in section "Introduction to Super Healing" for the proposed mechanism of super healing, the components of the fourth-rank healing tensor H is allowed to increase gradually beyond the values of the components of the fourth-rank identity tensor I, i.e., set $H_{ijkl} = (n + 1)\, I_{ijkl}$. In another way, previous relation can be written between the norms of the two fourth-rank tensors. The following expression can be obtained by substituting previous relation in Eq. 8 and simplifying

$$\overline{\sigma}_{ij} = \left[n\left(I_{ijkl} - M_{ijkl}^{-1}\right) + I_{ijkl}\right]^{-1} \sigma_{kl} \tag{9}$$

The process of anisotropic super healing is characterized by the expression obtained in Eq. 9. Based on Eq. 9, when the value of n tends to infinity, the value of the effective stress goes to zero. The same conclusion was obtained for super healing in the case of scalar damage and healing in previous sections. It should be noted that by implementing the appropriate constraints, the anisotropic super healing in Eq. 9 reduces to the scalar super healing in Eq. 3.

Damage/Healing and Super Healing in Plane Stress

Plane stress is solved to illustrate the processes of damage, healing, and super healing in this section. For this special case, the tensors of Eq. 8 are represented by vectors and matrices (Voyiadjis and Kattan 2006):

$$\{\sigma\} = \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} \tag{10a}$$

$$\{\overline{\sigma}\} = \begin{Bmatrix} \overline{\sigma}_{11} \\ \overline{\sigma}_{22} \\ \overline{\sigma}_{12} \end{Bmatrix} \tag{10b}$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{10c}$$

$$M = \frac{1}{\Delta}\begin{bmatrix} \psi_{22} & 0 & \varphi_{12} \\ 0 & \psi_{11} & \varphi_{12} \\ \dfrac{\varphi_{12}}{2} & \dfrac{\varphi_{12}}{2} & \dfrac{\psi_{11} + \psi_{22}}{2} \end{bmatrix} \tag{10d}$$

where $\psi_{11} = 1 - \varphi_{11}$ and $\psi_{22} = 1 - \varphi_{22}$. The denominator Δ in Eq. 10d is given by Voyiadjis and Kattan (2006):

$$\Delta = \psi_{11}\psi_{22} - \varphi_{12}^2 \tag{10e}$$

It should be noted that Eq. 10d is obtained based on the symmetrization procedure given by the following expression (Voyiadjis and Kattan 2006):

$$\overline{\sigma}_{ij} = \frac{1}{2}\left[\sigma_{ip}\left(\delta_{pj} - \varphi_{pj}\right)^{-1} + \left(\delta_{ip} - \varphi_{ip}\right)^{-1}\sigma_{pj}\right] \tag{10f}$$

The following 3×3 matrix representation can be written for the healing tensor of Eq. 8 for the case of plane stress:

$$H^{-1} = \begin{bmatrix} h_{11} & 0 & h_{12} \\ 0 & h_{22} & h_{12} \\ h_{12} & h_{12} & \dfrac{h_{11} + h_{22}}{2} \end{bmatrix} \tag{11}$$

Based on Eq. 11, the inverse healing tensor satisfies certain mathematical properties such that the components of this tensor are either positive or zero and the trace of this tensor $\left(h_{11} + h_{22} + \frac{h_{11}+h_{22}}{2}\right)$ is positive. Also, the norm of this tensor $\left(\sqrt{h_{11}^2 + h_{22}^2 + h_{12}^2}\right)$ is positive. Since the expression ${h_{11}}^2 + {h_{22}}^2 - {h_{12}}^2$ is not necessarily positive, this tensor is not necessarily positive definite.

It can be seen that the form of the inverse healing matrix which is postulated in Eq. 11 is similar to the form of the damage effect matrix in Eq. 10d. Substituting Eqs. 10f and 11 into Eq. 8 results in

$$\{\sigma\} = [X]\{\overline{\sigma}\} \tag{12}$$

where the components of the fourth-rank tensor $[X]$ are given by (the MATLAB Symbolic Math Toolbox is used to carry out the algebraic manipulations)

$$\begin{aligned} X_{11} = \big(&-2 + 3\varphi_{11} + \varphi_{22} - \varphi_{11}\varphi_{22} - \varphi_{11}^2 + \varphi_{12}^2 - 2h_{11}\varphi_{11} + h_{11}\varphi_{11}\varphi_{22} + h_{11}\varphi_{11}^2 \\ &- h_{11}\varphi_{12}^2 - 2h_{12}\varphi_{12} + 2h_{12}\varphi_{11}\varphi_{12}\big)/(\varphi_{11} + \varphi_{22} - 2) \end{aligned} \tag{13a}$$

$$X_{12} = \varphi_{12}(-\varphi_{12} + h_{22}\varphi_{12} - 2h_{12} + 2h_{12}\varphi_{11})/(\varphi_{11} + \varphi_{22} - 2) \tag{13b}$$

$$\begin{aligned} X_{13} = \big(&2\varphi_{12} - 2\varphi_{11}\varphi_{12} - 2h_{12}\varphi_{11} + h_{12}\varphi_{11}^2 + h_{12}\varphi_{11}\varphi_{22} - h_{11}\varphi_{12} - h_{22}\varphi_{12} \\ &+ h_{11}\varphi_{11}\varphi_{12} + h_{22}\varphi_{11}\varphi_{12}\big)/(\varphi_{11} + \varphi_{22} - 2) \end{aligned} \tag{13c}$$

$$X_{21} = \varphi_{12}(-\varphi_{12} + h_{11}\varphi_{12} - 2h_{12} + 2h_{12}\varphi_{22})/(\varphi_{11} + \varphi_{22} - 2) \tag{13d}$$

$$X_{22} = \big(-2 + \varphi_{11} + 3\varphi_{22} - \varphi_{11}\varphi_{22} - \varphi_{22}^2 + \varphi_{12}^2 - 2h_{22}\varphi_{22} + h_{22}\varphi_{11}\varphi_{22} + h_{22}\varphi_{22}^2 - h_{22}\varphi_{12}^2 - 2h_{12}\varphi_{12} + 2h_{12}\varphi_{22}\varphi_{12}\big)/(\varphi_{11} + \varphi_{22} - 2) \tag{13e}$$

$$X_{23} = \big(2\varphi_{12} - 2\varphi_{22}\varphi_{12} - 2h_{12}\varphi_{22} + h_{12}\varphi_{22}^2 + h_{12}\varphi_{11}\varphi_{22} - h_{11}\varphi_{12} - h_{22}\varphi_{12} + h_{11}\varphi_{22}\varphi_{12} + h_{22}\varphi_{22}\varphi_{12}\big)/(\varphi_{11} + \varphi_{22} - 2) \tag{13f}$$

$$X_{31} = \big(\varphi_{12} - \varphi_{11}\varphi_{12} - h_{12}\varphi_{11} - h_{12}\varphi_{22} - h_{11}\varphi_{12} + h_{11}\varphi_{11}\varphi_{12} + 2h_{12}\varphi_{11}\varphi_{22}\big)/(\varphi_{11} + \varphi_{22} - 2) \tag{13g}$$

$$X_{32} = \big(\varphi_{12} - \varphi_{22}\varphi_{12} - h_{12}\varphi_{11} - h_{12}\varphi_{22} - h_{22}\varphi_{12} + h_{22}\varphi_{22}\varphi_{12} + 2h_{12}\varphi_{11}\varphi_{22}\big)/(\varphi_{11} + \varphi_{22} - 2) \tag{13h}$$

$$X_{33} = \frac{1}{2}\big(-4 + 4\varphi_{11} + 4\varphi_{22} - 4\varphi_{11}\varphi_{22} - h_{11}\varphi_{11} - h_{11}\varphi_{22} - h_{22}\varphi_{11} - h_{22}\varphi_{22} - 4h_{12}\varphi_{12} + 2h_{12}\varphi_{11}\varphi_{12} + 2h_{12}\varphi_{22}\varphi_{12} + 2h_{11}\varphi_{11}\varphi_{22} + 2h_{22}\varphi_{11}\varphi_{22}\big)/ (\varphi_{11} + \varphi_{22} - 2) \tag{13i}$$

Special case, i.e., the case of principal components, is considered here for the sake of simplicity. For this case, by setting $\varphi_{12} = \varphi_{21} = 0$ and $h_{12} = h_{21} = 0$ in Eq. 13, the following expressions can be obtained:

$$X_{11} = \big(-2 + 3\varphi_{11} + \varphi_{22} - \varphi_{11}\varphi_{22} - \varphi_{11}^2 - 2h_{11}\varphi_{11} + h_{11}\varphi_{11}^2 + h_{11}\varphi_{11}\varphi_{22}\big)/(\varphi_{11} + \varphi_{22} - 2) \tag{14a}$$

$$X_{22} = \big(-2 + \varphi_{11} + 3\varphi_{22} - \varphi_{11}\varphi_{22} - \varphi_{22}^2 - 2h_{22}\varphi_{22} + h_{22}\varphi_{22}^2 + h_{22}\varphi_{11}\varphi_{22}\big)/(\varphi_{11} + \varphi_{22} - 2) \tag{14b}$$

$$X_{33} = \frac{1}{2}\big(-4 + 4\varphi_{11} + 4\varphi_{22} - 4\varphi_{11}\varphi_{22} - h_{11}\varphi_{11} - h_{11}\varphi_{22} - h_{22}\varphi_{11} - h_{22}\varphi_{22} + 2h_{11}\varphi_{11}\varphi_{22} + 2h_{22}\varphi_{11}\varphi_{22}\big)/(\varphi_{11} + \varphi_{22} - 2) \tag{14c}$$

It can be seen that based on Eq. 13 for this special case, all the other components of the fourth-rank tensor $[X]$ vanish. Simplifying Eq. 14 and substituting into Eq. 12 lead to obtain the following simple expressions (after some tedious algebraic manipulations):

$$\overline{\sigma}_{11} = \frac{\sigma_{11}}{1 - \varphi_{11}(1 - h_{11})} \tag{15a}$$

$$\overline{\sigma}_{22} = \frac{\sigma_{22}}{1 - \varphi_{22}(1 - h_{22})} \tag{15b}$$

$$\overline{\sigma}_{12} = \frac{\sigma_{12}}{1 - \left(1 - \frac{h_{11} + h_{22}}{2}\right)\left(\frac{\varphi_{11} + \varphi_{22} - 2\varphi_{11}\varphi_{22}}{\varphi_{11} + \varphi_{22} - 2}\right)} \tag{15c}$$

It can be seen from Eq. 15 that the principal equations of damage and healing (Eqs. 15a and 15b) reduce to the expression of the scalar case of Eq. 2 for the case of plane stress and the proposed super healing process is valid in the case of plane stress. This can be achieved by substituting values for the healing parameters h_{11} and h_{22} exceeding 1 and approaching a large number, and then the effective stress tends to zero. In addition, when these values tend to infinity, the elusive super material is obtained.

Characteristics of the Super Material

The use is made of the theories of super healing and undamageable materials to elucidate some of the characteristics of the sought super material within the continuum damage mechanics framework in this section. The concept of an undamageable material is proposed by Kobayashi (1992), Voyiadjis and Kattan (2012b), and Warren and Boettinger (1995). The value of the damage variable remains zero in this hypothetical type of material throughout the deformation process. Theoretically, such materials cannot be damaged. Constitutive equations of undamageable materials are derived through introducing a new type of material called the Voyiadjis–Kattan material of order n (Kobayashi 1992). This material type is a nonlinear elastic material which has a nonlinear strain energy form. Voyiadjis–Kattan materials of order n are based on higher-order strain energy forms that assume the general form

$$U = \frac{1}{2}\sigma\varepsilon^{n} \tag{16}$$

Nonlinear stress–strain relation for the given form of higher-order strain energy in Eq. 16 reads

$$\sigma = E\,\frac{1}{\varepsilon^{n}}\,e^{-2/\left[(n-1)\varepsilon^{(n-1)}\right]} \tag{17}$$

The reader is referred to Kobayashi (1992), Voyiadjis and Kattan (2012b), and Warren and Boettinger (1995) for full derivation and more details. This general form (Eq. 17) satisfies the initial conditions $\sigma = 0$ when $\varepsilon = 0$ since the limit of the expression for the stress tends to zero as the strain tends to zero. Equations 16 and 17 are valid for one-dimensional cases only. It can be concluded reasonably that

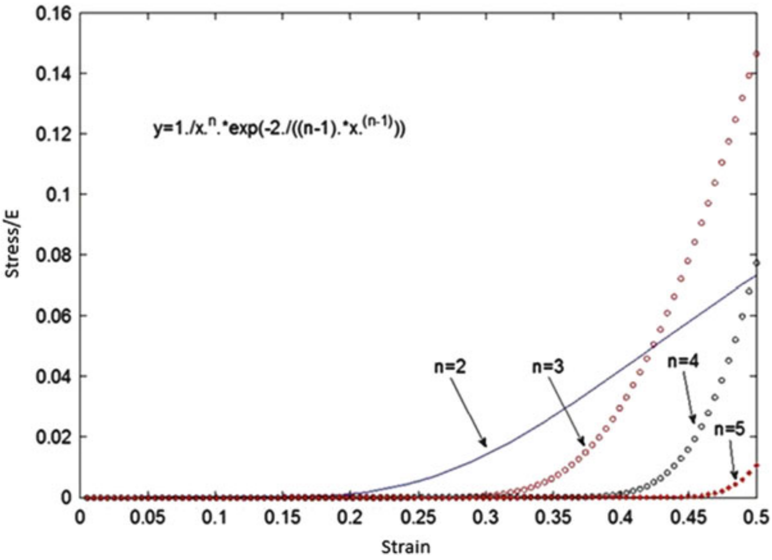

Fig. 8 Stress–strain curves based on Eq. 17 (After Voyiadjis and Kattan 2013a)

Voyiadjis–Kattan materials (and ultimately the undamageable material) can be manufactured in the future using a process based on the super healing model as outlined here. It can be seen that the Voyiadjis–Kattan material of order n is the same as a super-healed material of order n. Since the proposed higher-order strain energy forms (Eq. 16) admits integer values of the exponent n, it can be concluded that the super healing process also admits such integer values for n.

The following characteristics of the super material are outlined here based on the theory developed in sections "Introduction to Super Healing" and "Anisotropic Super Healing" in this chapter and the theory of undamageable materials as formulated by Gránásy et al. (2002), Kobayashi (1992), and Warren and Boettinger (1995).

The super material must be undamageable. Therefore, the properties of undamageable materials apply also to the super material. These properties are as outlined below. The value of the stress will remain equal to zero throughout the deformation process. The value of the damage variable will be equal to zero also throughout the deformation process. The super material has zero strain energy. This property is directly derived from above. The super material has nonzero strain values. Thus, the super material is a type of deformable material, not a rigid body. The super material is based on the proposed higher-order strain energy form of Eq. 16 taken in the limit when $n \to \infty$. The stress–strain relationship for the super material may be obtained from the elastic relation in Eq. 17 taken in the limit as $n \to \infty$. Some of the above items may be clearly deduced from the limit of Eq. 17. These characteristics are also clearly evident in Fig. 8 which was plotted based on Eq. 17.

Three Fundamental Issues in Continuum Damage Mechanics

In this section, the nature of the damage process within the framework of CDM is investigated by dissecting the expression of the effective stress into an infinite geometric series as follows:

$$ar + ar^2 + ar^3 + \ldots\ldots\ldots = \frac{a}{1-r} \tag{18}$$

The above geometric series is valid for $|r| < 1$. The effective stress given by

$$\overline{\sigma} = \frac{\sigma}{1-\varphi} \tag{19}$$

The effective stress given in Eq. 19 is the classical expression which is taken from the theory of CDM. In Eq. 19, φ is the damage variable (its value lies between zero and one), σ is the Cauchy stress, while $\overline{\sigma}$ is the corresponding effective stress. Comparing the right-hand side of the geometric series in Eq. 18 and the effective stress expression in Eq. 19 leads to the conclusion that the effective stress is equal to the sum of the infinite geometric series. It satisfies the condition since $0 < \varphi < 1$. Thus, by making analogy with the infinite geometric series in Eq. 18, the effective stress (Eq. 19) can be written as follows:

$$\overline{\sigma} = \sigma\left(1 + \varphi + \varphi^2 + \varphi^3 + \ldots\ldots\ldots\right) \tag{20}$$

Equation 20 is an infinite series exact relationship. It can be interpreted physically by considering the damage process as an infinite number of smaller damage processes or stages. Equation 20 can be rewritten in the following form since $\overline{\sigma}\overline{A} = \sigma A$, where A is the cross-sectional area and $\overline{A}$ is the effective cross-sectional area (the cross-sectional area in the fictitious effective configuration):

$$\frac{A}{\overline{A}} = 1 + \varphi + \varphi^2 + \varphi^3 + \ldots\ldots\ldots \tag{21}$$

Equation 21 shows that the damage process can be considered as the summation of several smaller damage processes or stages: the primary damage stage by taking the first two terms of the series, the secondary damage stage by taking the first three terms of the series, and the tertiary damage stage by taking the first four terms of the series. Although this process can be continued in an infinite number of smaller and smaller damage stages mathematically, but considering the first four terms of the infinite geometric series is sufficient for practical purposes.

Primary Damage Variable The first two terms of the series in Eq. 21 is considered to define the primary damage variable as follows:

$$\frac{A}{\overline{\overline{A}}} = 1 + \varphi_p \tag{22}$$

Equation 22 can be solved explicitly to obtain the following expression of the primary damage variable:

$$\varphi_p = \frac{A}{\overline{\overline{A}}} - 1 \tag{23}$$

Secondary Damage Variable The first three terms of the series in Eq. 21 are considered for defining the secondary damage variable as follows:

$$\frac{A}{\overline{\overline{A}}} = 1 + \varphi_S + \varphi_S^2 \tag{24}$$

The quadratic equation (Eq. 24) can be solved explicitly to obtain the following expression of the secondary damage variable:

$$\varphi_S = -\frac{1}{2} + \frac{1}{2}\sqrt{-3 + 4\frac{A}{\overline{\overline{A}}}} \tag{25}$$

Tertiary Damage Variable The first four terms of the series in Eq. 21 are considered to define the tertiary damage variable as follows:

$$\frac{A}{\overline{\overline{A}}} = 1 + \varphi_t + \varphi_t^2 + \varphi_t^3 \tag{26}$$

The cubic equation (Eq. 26) can be solved explicitly to obtain the following expression of the tertiary damage variable:

$$\begin{aligned} \varphi_t = & -\frac{1}{3} + \frac{1}{6}\sqrt[3]{-80 + 108\frac{A}{\overline{\overline{A}}} + 12\sqrt{48 - 120\frac{A}{\overline{\overline{A}}} + 81\left(\frac{A}{\overline{\overline{A}}}\right)^2}} \\ & - \frac{4}{3\sqrt[3]{-80 + 108\frac{A}{\overline{\overline{A}}} + 12\sqrt{48 - 120\frac{A}{\overline{\overline{A}}} + 81\left(\frac{A}{\overline{\overline{A}}}\right)^2}}} \end{aligned} \tag{27}$$

Thus, the explicit expressions for the damage variables at the primary, secondary, and tertiary damage stages have been established. In the next subsection, dissection of the damage process into the aforementioned three stages is presented mathematically.

Small Damage Processes

In this section, the problem of small damage processes is observed in details. In Voyiadjis and Mozaffari (2013), the following generalized relationship between the Cauchy stress and the effective stress is derived using the phase field method:

$$\overline{\sigma} = \frac{\sigma}{(1-\varphi)\sqrt{2\varphi+1}} \tag{28}$$

It is worth to mention that the effective stress expression given in Eq. 28 results in to a cubic formula for φ in terms of areas. The reader is referred to the work by authors for more details (Voyiadjis and Mozaffari 2013).

Equation 28 is compared with the classical expression given in Eq. 19. Considering the square root term that appears in the denominator of Eq. 28, the Taylor series expansion of the square root function is used to obtain the following approximation by taking the first two terms of the expansion:

$$\sqrt{2\varphi+1} \approx 1+\varphi \tag{29}$$

Equation 29 is valid for small values of φ, i.e., for small damage. The following expression for the effective stress of Eq. 28 can be written in the case of small damage:

$$\overline{\sigma} = \frac{\sigma}{(1-\varphi)(1+\varphi)} = \frac{\sigma}{1-\varphi^2} \tag{30}$$

The above formula (Eq. 30) corresponds to the damage variable $\varphi = \sqrt{\frac{A-\overline{A}}{A}}$ compared with $\varphi = \frac{A-\overline{A}}{A}$ for the classical case. Alternatively, the following expression for the effective stress is postulated in the case of large damage:

$$\overline{\sigma} = \frac{\sigma}{1-\sqrt{\varphi}} \tag{31}$$

which corresponds to the damage variable $\varphi = \left(\frac{A-\overline{A}}{A}\right)^2$

One can now generalize the two expressions for the effective stress (Eqs. 30 and 31) with an exponent n for the two different cases (small and large damage) when the increasing exponent goes from 2 to 3, 4, ... and ultimately to n where n tends to infinity. Therefore, the two following generalized definitions are proposed:

$$\overline{\sigma} = \frac{\sigma}{1-\varphi^n} \qquad \text{for small damage} \tag{32}$$

$$\overline{\sigma} = \frac{\sigma}{1-\varphi^{1/n}} \qquad \text{for large damage} \tag{33}$$

It can be seen that for normal (intermediate) damage then n = 1 in both cases.

The Concept of Undamageable Materials

In this section, previous issues are used to elaborate on the new undamageable material concept. These hypothetical materials were proposed recently by Voyiadjis and Kattan (2012b, 2013c, d) which can be compared with rubber materials (Arruda and Boyce 1993). Undamageable materials are compared with various nonlinear elastic materials taken from the book of Bower (2011) by Voyiadjis and Kattan (2012b, 2013c, d). Undamageable materials are assumed hypothetically to maintain a zero value for the damage variable through the loading process. Details of this formulation were presented by Voyiadjis and Kattan (2012b, 2013c, d) within the framework of CDM. Thus, it can be seen that undamageable materials are desirable since they cannot be damaged at all. It is hoped that the manufacturing technology will reach a stage in the future where the realization of this type of material can be achieved. The classical definition of the effective stress (Eq. 19) is modified to show that such materials maintain a zero value of the damage variable throughout the loading process as follows:

$$\overline{\sigma} = \frac{\sigma}{\sqrt[n]{1-\varphi}} \tag{34}$$

Performing the following derivation when n tends to infinity shows that the stress and the effective stress are equal in undamageable materials:

$$\overline{\sigma} = \frac{\sigma}{\sqrt[n]{1-\varphi}} = \frac{\sigma}{(1-\varphi)^{1/n}} = \frac{\sigma}{(1-\varphi)^{1/\infty}} = \frac{\sigma}{(1-\varphi)^{0}} = \frac{\sigma}{1} = \sigma \tag{35}$$

Thus, one obtains the undamageable material in this case. In their previous publications (Voyiadjis and Kattan 2012b, 2013c, d), they presented the concept of undamageable materials using the definition of the damage variable in terms of elastic stiffness degradation. The formulation is now supported further by presenting the concept of undamageable materials using a slightly modified form of the effective stress based on the cross-sectional area reduction as shown in Eqs. 17 and 18.

Internal Damage Processes Leading to a Singularity in a Continuous Region

Providing a possible link between the subjects of damage mechanics and fracture mechanics is the aim of this section. Phenomenological study of the various internal damage mechanisms without emphasis on the actual geometry of micro-cracks, micro-voids, or other micro-defects is the main subject in continuum damage mechanics. The study of the various forms of crack propagation and coalescence in great details but without much discussion on initiation of these defects is the main subject in fracture mechanics. Usually some form of energy threshold is used to indicate these defects initiation but without showing precisely the way that they

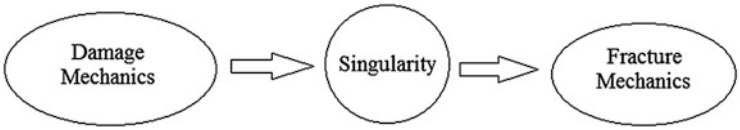

Fig. 9 The singularity could provide a crucial link between damage mechanics and fracture mechanics (After Voyiadjis and Kattan 2013b)

form. Recently, numerical simulations like the finite element method are used by some researchers to show how cracks initiate in solids (Yu et al. 2012). Currently there is no analytical closed-form solution for the precise method in which a singularity arises in a continuous region.

In this section and within the framework of continuum damage mechanics, a proposed sequence of internal damage processes within a continuous region is postulated and described mathematically. It is shown that this sequence of internal damage processes lead to a singularity in the continuous region. The resulting singularity could be interpreted in several ways. It could represent the crack tip of a forming micro-crack, the tip of a forming micro-void, or the tip of any other forming micro-defect. This emerging singularity could also provide a crucial link between the subjects of damage mechanics and fracture mechanics (see Fig. 9).

Mathematical Formulation

In this section, the principles of continuum damage mechanics are used to show how a singularity arises in a continuous region. The cross-sectional area of the damaged material is shown by A, while the corresponding damaged area is shown by A_o. Based on continuum damage mechanics, the following classic equation is common:

$$\phi_o = \frac{A_o}{A} \tag{36}$$

The force on the cross-sectional area A is equal to σA, while the force on the undamaged area $A - A_0$ is equal to $\overline{\sigma}_0(A - A_0)$. Thus, the following equation shows the equality of the forces on both damaged material (real configuration) and undamaged material (fictitious configuration):

$$\sigma A = \overline{\sigma}_0(A - A_0) \tag{37a}$$

Equating Eqs. 36 and 37a and simplifying lead to

$$\sigma = \overline{\sigma}_o(1 - \phi_o) \tag{37b}$$

where ϕ_o is the damage variable, σ is the stress in the damaged configuration, and $\overline{\sigma}_o$ is the effective stress that is associated with ϕ_o in the fictitious effective (undamaged) configuration (see Fig. 10).

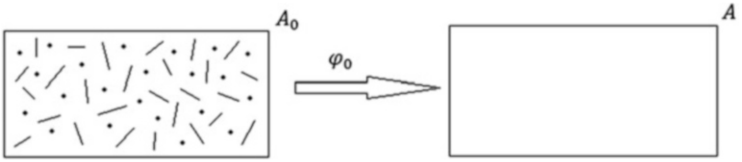

Fig. 10 The damaged state and fictitious undamaged state (After Voyiadjis and Kattan 2013b)

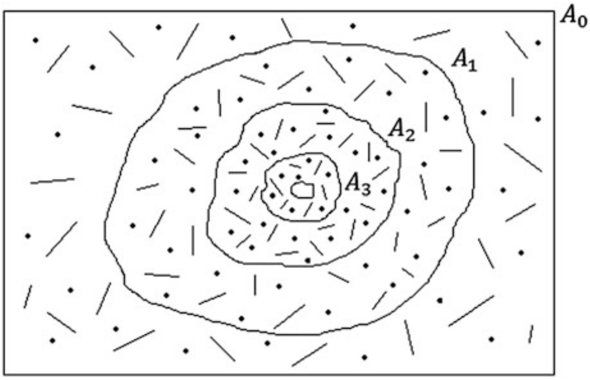

Fig. 11 The sequence of decreasing subareas leading to the singularity (After Voyiadjis and Kattan 2013b)

A subarea A_1 of the damaged area A_o where $A_1 < A_o$ (see Fig. 11) is considered and a new damage variable ϕ_1 acting on this subarea A_1 with further damage is defined as follows (in this way, A_1 is subjected to ϕ_o then followed by ϕ_1):

$$\phi_1 = \frac{A_1}{A} \tag{38}$$

The force on the cross-sectional area A is given by σA, and the forces on the undamaged areas $A - A_0$ and $A - A_1$ are given by $\overline{\sigma}_0(A - A_0)$ and $\overline{\sigma}_1(A - A_1)$, respectively. Therefore, the following equation can be written based on the force equality on both damaged and undamaged configurations:

$$\sigma A = \overline{\sigma}_0(A - A_0) + \overline{\sigma}_1(A - A_1) \tag{39a}$$

Substituting Eqs. 36 and 38 into Eq. 39a and simplifying result in

$$\sigma = \overline{\sigma}_0(1 - \phi_o) + \overline{\sigma}_1(1 - \phi_1) \tag{39b}$$

Clearly $\phi_o > \phi_1$ where ϕ_1 is a new damage variable defined on the subarea A_1 and $\overline{\sigma}_1$ is the effective stress associated with ϕ_1.

Consequently, a subarea A_2 of the damaged area A_1 where $A_2 < A_1$ (see Fig. 11) is considered and a new damage variable ϕ_2 acting on this subarea A_2 with further damage is defined (in this way, A_2 is subjected to ϕ_o then followed by ϕ_1, then finally followed by ϕ_2). Therefore, new damage variable reads

$$\phi_2 = \frac{A_2}{A} \tag{40}$$

The force on the cross-sectional area A is obtained as $\sigma\, A$, and the forces on the undamaged areas $A - A_0$, $A - A_1$, and $A - A_2$ are obtained by $\overline{\sigma}_0(A - A_0)$, $\overline{\sigma}_1(A - A_1)$, and $\overline{\sigma}_2(A - A_2)$, respectively. Therefore, the following equation can be written based on the force equality on both damaged and undamaged configurations:

$$\sigma A = \overline{\sigma}_0(A - A_0) + \overline{\sigma}_1(A - A_1) + \overline{\sigma}_2(A - A_2) \tag{41a}$$

Substituting Eqs. 36, 38, and 40 into Eq. 41a and simplifying lead to

$$\sigma = \overline{\sigma}_0(1 - \phi_o) + \overline{\sigma}_1(1 - \phi_1) + \overline{\sigma}_2(1 - \phi_2) \tag{41b}$$

Clearly $\phi_1 > \phi_2$ where ϕ_2 is a new damage variable defined on the subarea A_2 and $\overline{\sigma}_2$ is the effective stress associated with ϕ_2.

This process can be continued by defining n subareas $A_1 > A_2 > \ldots\ldots > A_n$ along with n damage variables $\phi_1 > \phi_2 > \ldots\ldots > \phi_n$.

Thus, a strictly decreasing monotonic sequence of damage variables is obtained as follows: $\phi_o, \phi_1, \phi_2, \ldots\ldots, \phi_n$.

The above sequence converges to a limit based on the mathematics of sequences and series since the value of each single damage variable is less than 1. Thus, the following convergent series can be written as

$$\phi_o + \phi_1 + \phi_2 + \phi_3 + \ldots\ldots + \phi_n + \ldots\ldots\ldots = \phi \tag{42}$$

where ϕ is the limit of the sequence and sum of the series.

Based on a direct extension of Eqs. 37b, 39b, and 41b, the following equation for the stress can be written as

$$\sigma = \overline{\sigma}_0(1 - \phi_o) + \overline{\sigma}_1(1 - \phi_1) + \ldots\ldots. + \overline{\sigma}_n(1 - \phi_n) + \ldots\ldots\ldots. \tag{43}$$

The question that arises is what happens when the above sequence tend to infinity. Based on Eq. 48, at infinity the stress becomes infinite, while the damage variable ϕ_n becomes zero as $n \to \infty$ and the subarea A_n collapses to a point, i.e., the sought after singularity.

Example: Special Case A special case is discussed to illustrate the above concept. The following equations are valid assuming that the successive damage variables in Eq. 42 are related by a constant ratio α:

$$\phi_1 = \alpha\phi_o \tag{44}$$

$$\phi_2 = \alpha\phi_1 \tag{45}$$

where $0 < \alpha < 1$. Thus, Eq. 42 becomes

$$\phi_o\left(1 + \alpha + \alpha^2 + \alpha^3 + \ldots\ldots + \alpha^{n-1}\right) = \phi \tag{46}$$

The expression inside the parenthesis in Eq. 46 is a geometric series that sums to the value $\frac{1-\alpha^n}{1-\alpha}$. Thus, Eq. 46 can be written as

$$\phi_o \frac{1-\alpha^n}{1-\alpha} = \phi \tag{47}$$

Considering an infinite geometric series, i.e., when $n \to \infty$, then $\alpha^n \to 0$, since $0 < \alpha < 1$. Thus, Eq. 47 simplifies to the following form:

$$\frac{\phi_o}{1-\alpha} = \phi \tag{48}$$

And since the value of ϕ is less than 1, then one obtains the constraint $\alpha < 1 - \phi_o$ on the value of the constant parameter α from Eq. 48. Therefore, the value of the damage variable ϕ at the singularity in terms of the initial value of the damage variable ϕ_o can be obtained from Eq. 48.

Conclusion

In this chapter, a new type of healing/strengthening process in materials called super healing is proposed along with a new hypothetical type of material that is called the super material. The mechanics of the super healing process has been outlined using both scalar variables and anisotropic tensors. In addition, the mechanics of scalar damage/healing is reviewed and elaborated on the mechanics of anisotropic damage/healing. For further clarification and illustration of these new concepts, the special case of plane stress was solved. Finally and in order to elucidate the characteristics of the sought super material, it is concluded that the super material has to be undamageable within the theory of elastic undamageable elastic materials framework.

The authors did not present the physical and metallurgical aspects of this theory in this work but only the theoretical mathematical formulation. This is because it is not clear yet to the authors how these types of advanced materials could be manufactured. It is hoped that the authors would be able to address the physical and metallurgical aspects in forthcoming work. The authors reiterate their viewpoint that this mathematical formulation lays a possible groundwork for any future development in this regard. The authors are still hopeful that some form of strengthened material may be realized in the near future. The various figures and equations presented here should form the basis of the future technology of undamageable materials that will be effectively indestructible.

The final question is what could be the interest (or practical use) of a material that, even if undamageable, would deliver an extremely high valued strain under a totally negligible stress (as demonstrated on Fig. 8). The answer is that as the exponent n approaches infinity, the material becomes very soft and is not ideal for structural behavior related to practical applications. The issue is that at some value, this exponent will be presenting a reasonable stiffness for structural applications while maintaining a high fidelity for an undamaged material response. This is termed a super material in structural applications. It is also noted that in reality, the exponent n can never reach infinity in real applications. For practical applications, a finite value of n can be used although it may be very high.

It is shown that by considering a sequence of internal damage processes such that the value of the damage variable in each process is less than that in the preceding process leads to the sequence convergence to a finite value of damage along with the fact that the sequence is a strictly decreasing monotonic sequence. It is also shown considering such an infinite sequence leads to the infinite sequence of subareas which converges to a point at infinity. This emerging point can be considered as the sought singularity. This singularity may be visualized as the crack tip of an emerging micro-crack, the tip of an emerging micro-void, or the tip of any other emerging micro-defect. It is postulated that a singularity-like point will be reached at a finite but large enough value of the parameter n since the point at infinity may never be reached in practical problems. Consequently, micro-cracks, micro-voids, and other micro-defects initiate in continuous regions out of nowhere. A possible sequence of internal damage mechanisms that could produce a singularity is illustrated in the last section of this chapter which may be considered as a crucial link between the subjects of damage mechanics and fracture mechanics.

References

J.A. Adam, A simplified model of wound healing (with particular reference to the critical size defect). Math. Comput. Model. **30**, 23–32 (1999)

E.M. Arruda, M.C. Boyce, A three-dimensional constitutive model for the large stretch behavior of rubber elastic materials. J. Mech. Phys. Solids **41**, 389–412 (1993)

E.J. Barbero, F. Greco, P. Lonetti, Continuum damage-healing mechanics with application to self-healing composites. Int. J. Damage Mech. **14**, 51–81 (2005)

A.F. Bower, *Applied Mechanics of Solids* (CRC press, Boca Raton, 2011)

A. Cauvin, R.B. Testa, Damage mechanics : basic variables in continuum theories. Int. J. Solids Struct. **36**, 747–761 (1999)

J.L. Chaboche, On some modifications of kinematic hardening to improve the description of ratchetting effects. Int. J. Plast. **7**, 661–678 (1991)

J.L. Chaboche, Thermodynamic formulation of constitutive equations and application to the viscoplasticity and viscoelasticity of metals and polymers. Int. J. Solids Struct. **34**, 2239–2254 (1997)

C.L. Chow, M. Jie, Anisotropic damage-coupled sheet metal forming limit analysis. Int. J. Damage Mech. **18**, 371–392 (2009)

C. Chow, J. Wang, An anisotropic theory of elasticity for continuum damage mechanics. Int. J. Fract. **33**, 3–16 (1987)

W.L. George, J.A. Warren, A parallel 3D dendritic growth simulator using the phase-field method. J. Comput. Phys. **177**, 264–283 (2002)

V. Ginzburg, On the theory of superconductivity. Il Nuovo Cimento (1955–1965) **2**, 1234–1250 (1955)

V. Ginzburg, L.D. Landau, On the theory of superconductivity. Zh. Eksp. Teor. Fiz. **20**(1950), 1064–1082 (1965). Translation in Collected papers of L.D.Landau. Pergamon, Oxford

L. Gránásy, T. Börzsönyi, T. Pusztai, Nucleation and bulk crystallization in binary phase field theory. Phys. Rev. Lett. **88**, 206105 (2002)

N.R. Hansen, H.L. Schreyer, A thermodynamically consistent framework for theories of elastoplasticity coupled with damage. Int. J. Solids Struct. **31**, 359–389 (1994)

M. John, G. Li, Self-healing of sandwich structures with grid stiffened shape memory polymer syntactic foam core. Smart Mater. Struct. **19**, 1–12 (2010)

L.M. Kachanov, On the creep fracture time. Izv Akad. Nauk USSR Otd. Tekh. **8**, 26–31 (1958)

P.I. Kattan, G.Z. Voyiadjis, A plasticity-damage theory for large deformation of solids—II. Applications to finite simple shear. Int. J. Eng. Sci. **31**, 183–199 (1993)

P.I. Kattan, G.Z. Voyiadjis, Decomposition of damage tensor in continuum damage mechanics. J. Eng. Mech. **127**, 940–944 (2001)

R. Kobayashi, Simulations of three dimensional dendrites, in *Pattern Formation in Complex Dissipative Systems*, ed. by S. Kai (World Scientific, Singapore, 1992), pp. 121–128

L.D. Landau, D. Ter Haar, *Collected Papers of LD Landau* (Pergamon Press, Oxford, 1965)

H. Lee, K. Peng, J. Wang, An anisotropic damage criterion for deformation instability and its application to forming limit analysis of metal plates. Eng. Fract. Mech. **21**, 1031–1054 (1985)

J. Lemaitre, How to use damage mechanics. Nucl. Eng. Des. **80**, 233–245 (1984)

J. Lemaitre, Coupled elasto-plasticity and damage constitutive equations. Comput. Methods Appl. Mech. Eng. **51**, 31–49 (1985)

J. Lemaitre, J.L. Chaboche, *Mechanics of Solid Materials* (Cambridge University Press, Cambridge, 1990)

G. Li, M. John, A self-healing smart syntactic foam under multiple impacts. Compos. Sci. Technol. **68**, 3337–3343 (2008)

G. Li, V.D. Muthyala, Impact characterization of sandwich structures with an integrated orthogrid stiffened syntactic foam core. Compos. Sci. Technol. **68**, 2078 (2008)

G. Li, D. Nettles, Thermomechanical characterization of a shape memory polymer based self-repairing syntactic foam. Polymer **51**, 755–762 (2010)

G. Li, N. Uppu, Shape memory polymer based self-healing syntactic foam: 3-D confined thermomechanical characterization. Compos. Sci. Technol. **70**, 1419–1427 (2010)

Y.L. Liu, Y.W. Chen, Thermally reversible cross-linked polyamides with high toughness and self-repairing ability from maleimide- and furan-functionalized aromatic polyamides. Macromol. Chem. Phys. **208**, 224–232 (2007)

I. Loginova, G. Amberg, J. Ågren, Phase-field simulations of non-isothermal binary alloy solidification. Acta Mater. **49**, 573–581 (2001)

V.A. Lubarda, D. Krajcinovic, Damage tensors and the crack density distribution. Int. J. Solids Struct. **30**, 2859–2877 (1993)

S. Miao, M.L. Wang, H.L. Schreyer, Constitutive models for healing of materials with application to compaction of crushed rock salt. J. Eng. Mech. **121**, 1122–1129 (1995)

S. Murakami, Notion of continuum damage mechanics and its application to anisotropic creep damage theory. ASME Trans. J. Eng. Mater. Technol. **105**, 99–105 (1983)

B. Murray, A. Wheeler, M. Glicksman, Simulations of experimentally observed dendritic growth behavior using a phase-field model. J. Cryst. Growth **154**, 386–400 (1995)

M. Naderi, A. Kahirdeh, M. Khonsari, Dissipated thermal energy and damage evolution of Glass/Epoxy using infrared thermography and acoustic emission. Compos. Part B **43**, 1613–1620 (2012)

S. Nemat-Nasser, Decomposition of strain measures and their rates in finite deformation elastoplasticity. Int. J. Solids Struct. **15**, 155–166 (1979)

S. Nemat-Nasser, *On Finite Plastic Flow of Crystalline Solids and Geomaterials* (DTIC Document, Fort Belvoir, 1983)

J. Nji, G. Li, A self-healing 3D woven fabric reinforced shape memory polymer composite for impact mitigation. Smart Mater. Struct. **19**, 1–9 (2010a)

J. Nji, G. Li, A biomimic shape memory polymer based self-healing particulate composite. Polymer **51**, 6021–6029 (2010b)

J.W.C. Pang, I.P. Bond, A hollow fibre reinforced polymer composite encompassing self-healing and enhanced damage visibility. Compos. Sci. Technol. **65**, 1791–1799 (2005)

T. Park, G. Voyiadjis, Kinematic description of damage. J. Appl. Mech. **65**, 93–98 (1998)

R. Pavan, B. Oliveira, S. Maghous, G. Creus, A model for anisotropic viscoelastic damage in composites. Compos. Struct. **92**, 1223–1228 (2010)

Y.N. Rabotnov, Paper 68: on the equation of state of creep, in *Proceedings of the Institution of Mechanical Engineers, Conference Proceedings*, SAGE Publications, 1963, pp. 2–117, 112–122

Y.N. Rabotnov, Creep rupture, in *Proceedings of the XII International Congress on Applied Mechanics*, 1968, pp. 342–349

F. Sidoroff, Description of anisotropic damage application to elasticity, in *Proceedings of the IUTAM Colloquium on Physical Nonlinearities in Structural Analysis*, Berlin, 1981, pp. 237–244

A.H.R.W. Simpson, T.N. Gardner, M. Evans, J. Kenwright, Stiffness, strength and healing assessment in different bone fractures – a simple mathematical model. Injury **31**, 777–781 (2000)

I. Singer-Loginova, H. Singer, The phase field technique for modeling multiphase materials. Rep. Prog. Phys. **71**, 106501 (2008)

K.S. Toohey, N.R. Sottos, J.A. Lewis, J.S. Moore, S.R. White, Self-healing materials with microvascular networks. Nat. Mater. **6**, 581–585 (2007)

J.D. van der Waals, The thermodynamic theory of capillarity under the hypothesis of a continuous variation of density. J. Stat. Phys. **20**, 200–244 (1979)

R.J. Varley, S. van der Zwaag, Towards an understanding of thermally activated self-healing of an ionomer system during ballistic penetration. Acta Mater. **56**, 5737–5750 (2008)

G.Z. Voyiadjis, Degradation of elastic modulus in elastoplastic coupling with finite strains. Int. J. Plast. **4**, 335–353 (1988)

G.Z. Voyiadjis, P.I. Kattan, A coupled theory of damage mechanics and finite strain elasto-plasticity—II. Damage and finite strain plasticity. Int. J. Eng. Sci. **28**, 505–524 (1990)

G.Z. Voyiadjis, P.I. Kattan, A plasticity-damage theory for large deformation of solids—I. Theoretical formulation. Int. J. Eng. Sci. **30**, 1089–1108 (1992)

G.Z. Voyiadjis, P.I. Kattan, *Damage Mechanics* (CRC, Boca Raton, 2005)

G.Z. Voyiadjis, P.I. Kattan, *Advances in Damage Mechanics: Metals and Metal Matrix Composites With an Introduction to Fabric Tensors* (2nd edition), 742 p., (Elsevier, Oxford, ISBN: 0-08-044688-4, 2006)

G.Z. Voyiadjis, P.I. Kattan, A comparative study of damage variables in continuum damage mechanics. Int. J. Damage Mech. **18**, 315–340 (2009)

G.Z. Voyiadjis, P.I. Kattan, Mechanics of damage processes in series and in parallel: a conceptual framework. Acta Mech. **223**, 1863–1878 (2012a)

G.Z. Voyiadjis, P.I. Kattan, A new class of damage variables in continuum damage mechanics. J. Eng. Mater. Technol. **134**, 021016 (2012b)

G.Z. Voyiadjis, P.I. Kattan, Healing and Super Healing in Continuum Damage Mechanics. Int. J. Damage Mech. 23(2), 245–260 (2014)

G.Z. Voyiadjis, P.I. Kattan, How a Singularity Forms in Continuum Damage Mechanics. Mech. Res. Commun. **55**, 86-88 (2014)

G.Z. Voyiadjis, P.I. Kattan, Introduction to the mechanics and design of undamageable materials. Int. J. Damage Mech. **22**, 323–335 (2013c)

G.Z. Voyiadjis, P.I. Kattan, On the theory of elastic undamageable materials. J. Eng. Mater. Technol. **135**, 021002 (2013d)

G.Z. Voyiadjis, N. Mozaffari, Nonlocal damage model using the phase field method: theory and applications. Int. J. Solids Struct. **50**, 3136–3151 (2013)
G.Z. Voyiadjis, T. Park, Local and interfacial damage analysis of metal matrix composites. Int. J. Eng. Sci. **33**, 1595–1621 (1995)
G.Z. Voyiadjis, T. Park, Anisotropic damage for the characterization of the onset of macro-crack initiation in metals. Int. J. Damage Mech. **5**, 68–92 (1996)
G. Voyiadjis, T. Park, Anisotropic damage effect tensors for the symmetrization of the effective stress tensor. J. Appl. Mech. **64**, 106–110 (1997)
G.Z. Voyiadjis, Z.N. Taqieddin, P.I. Kattan, Theoretical formulation of a coupled elastic—plastic anisotropic damage model for concrete using the strain energy equivalence concept. Int. J. Damage Mech. **18**, 603–638 (2009)
G.Z. Voyiadjis, A. Shojaei, G. Li, P. Kattan, Continuum damage-healing mechanics with introduction to new healing variables. Int. J. Damage Mech. **21**(3), 391–414 (2012)
G.Z. Voyiadjis, A. Shojaei, G. Li, A generalized coupled viscoplastic–viscodamage–viscohealing theory for glassy polymers. Int. J. Plast. **28**, 21–45 (2012)
S.-L. Wang, R.F. Sekerka, Algorithms for phase field computation of the dendritic operating state at large supercoolings. J. Comput. Phys. **127**, 110–117 (1996)
J.A. Warren, W.J. Boettinger, Prediction of dendritic growth and microsegregation patterns in a binary alloy using the phase-field method. Acta Metall. Mater. **43**, 689–703 (1995)
A. Wheeler, B. Murray, R. Schaefer, Computation of dendrites using a phase field model. Phys. D Nonlinear Phenom. **66**, 243–262 (1993)
S.R. White, N.R. Sottos, P.H. Geubelle, J.S. Moore, M.R. Kessler, S.R. Sriram, E.N. Brown, S. Viswanathan, Autonomic healing of polymer composites. Nature **409**, 794–797 (2001)
H.-L. Yu, C. Lu, K. Tieu, G.-Y. Deng, A numerical model for simulation of crack initiation around inclusion under tensile load. J. Comput. Theor. Nanosci. **9**, 1745–1749 (2012)
K. Yuan, J. Ju, New strain energy–based coupled elastoplastic damage-healing formulations accounting for effect of matric suction during earth-moving processes. J. Eng. Mech. **139**, 188–199 (2012)
F. Zaïri, M. Naït-Abdelaziz, J.-M. Gloaguen, J.-M. Lefebvre, A physically-based constitutive model for anisotropic damage in rubber-toughened glassy polymers during finite deformation. Int. J. Plast. **27**, 25–51 (2011)
M. Zako, N. Takano, Intelligent material systems using epoxy particles to repair microcracks and delamination damage in GFRP. J. Intell. Mater. Syst. Struct. **10**, 836–841 (1999)