Introduction to Laboratory 2

Laboratory 2 concerns the implementation of the **Newton-Raphson method**, which is an iterative method to find the root of a function (i.e. to find the x_{Root} value for which $f(x_{Root}) = 0$). In the Newton-Raphson method we start with an initial value x_0 (this value should be sufficiently close to x_{Root} in order for the method to converge properly, see examples in the exercise 2). By employing the values of the function f and of its derivative f' at x_0, one can calculate the tangent line that crosses the x axis at the point x_1 (see Figure 1). Then, the value x_1 provides a new approximation to the root, which can be employed to calculate a new tangent line crossing the x axis at the point x_2. Next, the process should be iterated until convergence is obtained.

The tangent line is a linear function of general form $y(x) = Ax + B$. The slope of the first tangent line (i.e. at x_0) is given by $A = f'(x_0)$. Moreover, at the point x_1 we have $y(x_1) = 0 = Ax_1 + B = f'(x_0)x_1 + B$. This means that $B = -f'(x_0)x_1$. Therefore, the equation for the first tangent line is given by

$$y(x) = f'(x_0)(x - x_1)$$

At the point x_0 we have $y(x_0) = f(x_0) = f'(x_0)(x_0 - x_1)$.

This leads to $f(x_0)/f'(x_0) = x_0 - x_1$, and finally to an equation relating x_0 to x_1:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

The procedure can be iterated. Thus, a new approximation to the root x_{i+1} is calculated from the previous approximation x_i according to the equation:

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

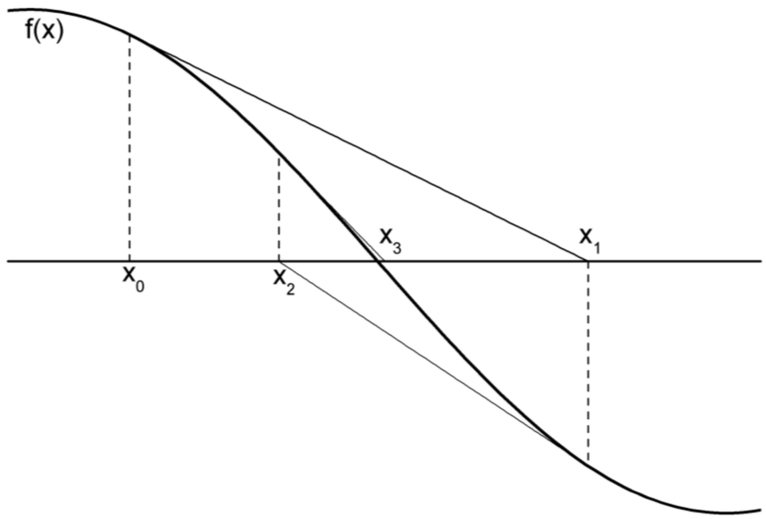

Figure 1: Example illustrating the Newton-Raphson method.

Solutions:

1) For the function $f(x) = \cos x - x$ and its derivative $f'(x) = -\sin x - 1$, starting with the initial value $x_0 = 0$ and *Tolerance*=10^{-8}, the program does 5 iterations. The root x_0 is equal to 0.739 085 13. In the previous laboratory we found that the Bisection method needs 28 iterations to converge. Therefore, we see that the Newton-Raphson method is faster than the Bisection method. However, the Newton-Raphson method requires additionally the derivative $f'(x)$, which is not needed in the Bisection method.

2) The four roots of the polynomial $P(x)$ are 0.183 434 64 ; 0.525 532 41 ; 0.796 666 48 ; 0.960 289 86.

A problem can happen with the Newton-Raphson method when we reach a point close to an extremum of $P(x)$, at which $f'(x) \approx 0$. In this case, the division by $f'(x_i)$ in the equation $x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$ leads to a new approximation x_{i+1} much further to the root than the previous approximation x_{i+1}. At the limit when $f'(x_i) \to 0$, the Newton-Raphson method totally fails because $x_{i+1} \to \pm\infty$.

An example of such a behavior is obtained for the initial value $x_0 = 0.4$. In this case, one would expect the method to converge to the nearest root at 0.525 532 41, but instead it converges to the root at 0.960 289 86.

3) The relation $x = 13^{2/3}$ can be re-written as $x^3 = 13^2 = 169$, and finally as $x^3 - 169 = 0$. By defining $f(x) = x^3 - 169$, one can use the Newton-Raphson method to calculate x_{Root} for which $f(x_{Root}) = 0$. The results is 5.528 774 81.